

Infinite-dimensional reductive monoids associated to highest weight representations of Kac-Moody groups

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Abstract

Starting with a highest weight representation of a Kac-Moody group over the complex numbers, we construct a monoid whose unit group is the image of the Kac-Moody group under the representation, multiplied by the nonzero complex numbers. We show that this monoid has similar properties to those of a \mathcal{J} -irreducible reductive linear algebraic monoid. In particular, the monoid is unit regular and has a Bruhat decomposition, and the idempotent lattice of the generalized Renner monoid of the Bruhat decomposition is isomorphic to the face lattice of the convex hull of the Weyl group orbit of the highest weight.

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1 Introduction

Let \mathbf{G} be a semisimple linear algebraic group over \mathbb{C} . Let $\rho : \mathbf{G} \rightarrow GL(V)$ be an irreducible rational representation of \mathbf{G} , i.e., an irreducible highest weight representation of \mathbf{G} with dominant highest weight μ . Then the Zariski closure

$$M(\rho) := \overline{\mathbb{C}^\times \rho(\mathbf{G})} \subseteq \text{End}(V) \quad (1)$$

is a reductive linear algebraic monoid with reductive unit group $\mathbb{C}^\times \rho(\mathbf{G})$. It belongs to the class of \mathcal{J} -irreducible reductive linear algebraic monoids [35, Proposition 4.2]. It is normal if and only if μ is minuscule [6, Theorem 3.1]. The most familiar example is $M(m+1, \mathbb{C})$, which can be obtained from $SL(m+1, \mathbb{C})$ and its natural representation on \mathbb{C}^{m+1} . Here the corresponding highest weight is the first fundamental dominant weight, which is minuscule.

The theory of reductive linear algebraic monoids has been developed mainly by L. E. Renner and M. S. Putcha. Contributions to the theory have been made also by several other people, for example M. Brion, S. Doty, W. Huang, J. Okniński, A. Rittatore, L. Solomon, and E. B. Vinberg. Excellent accounts can be found in [34, 37, 39]. Much of the information about a reductive linear algebraic monoid is encoded in its combinatorial objects such as cross-section lattice, type functions, and Renner monoid which is a finite unit regular monoid whose unit group is the Weyl group and whose idempotent lattice is isomorphic to the face lattice of a certain polyhedral cone. In particular, \mathcal{J} -irreducible

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reductive linear algebraic monoids are quite accessible because here the idempotent lattice of a Renner monoid is isomorphic to the face lattice of the convex hull of a single Weyl group orbit. They have been further investigated in [19]–[22]; some special \mathcal{J} -irreducible reductive linear algebraic monoids, referred to as classical algebraic monoids, have been studied in [16, 17, 18].

To construct the monoids in (1) we may restrict to semisimple simply connected linear algebraic groups. Now the minimal Kac-Moody groups as defined in [15, 33], which we simply call Kac-Moody groups, generalize semisimple simply connected linear algebraic groups. Is it possible to carry out a similar construction for these groups and their irreducible highest weight representations with dominant highest weights?

There are differences and obstructions. These Kac-Moody groups generalize semisimple simply connected linear algebraic groups only as groups. They are constructed by representation and group theoretical methods and do not carry a coordinate ring or even a topology by their construction. The common theory to deal with finite- as well as infinite-dimensional algebraic geometric objects is the theory of schemes, but a more elementary approach would be appropriate in our context. It is also worth to note the following difference. If the generalized Cartan matrix is degenerate then the Kac-Moody group contains a nontrivial maximal central torus. To descend to the factor group is not a reasonable option because many of the highest weight representation would be lost. For example, an affine Kac-Moody group factored by its maximal central torus has only one-dimensional highest weight representations.

Let \mathbf{G} be a Kac-Moody group over \mathbb{C} , and let $\rho : \mathbf{G} \rightarrow \mathrm{GL}(V)$ be a highest weight representation with dominant highest weight μ . To generalize the construction (1) we proceed as follows:

- (1) We define the monoid $M(\rho)$ algebraically and show that it has similar algebraic properties to those of a \mathcal{J} -irreducible reductive linear algebraic monoid.
- (2) We equip $M(\rho)$ with a coordinate ring and show that $M(\rho)$ has similar algebraic geometric properties to those of a \mathcal{J} -irreducible reductive linear algebraic monoid. In general, $M(\rho)$ is infinite-dimensional. We determine its Lie algebra.
- (3) We provide a certain subalgebra of $\mathrm{End}(V)$, which contains $\mathbb{C}^\times \rho(\mathbf{G})$, with a coordinate ring and show that $M(\rho) = \overline{\mathbb{C}^\times \rho(\mathbf{G})}$.

This requires some amount of work. In this article we carry out part (1). In a subsequent article parts (2) and (3) are treated. We also restrict to the case where no indecomposable component of \mathbf{G} stabilizes the highest weight space V_μ , which captures already all relevant ideas and is less technical to write down.

Let (\mathbf{B}, \mathbf{N}) be the BN-pair obtained by the construction of the Kac-Moody group \mathbf{G} . Let $W := \mathbf{N}/\mathbf{T}$, where $\mathbf{T} := \mathbf{B} \cap \mathbf{N}$, be the associated Weyl group with a set S of simple reflections. To keep our notation simple we set

$$G := \mathbb{C}^\times \rho(\mathbf{G}), \quad B := \mathbb{C}^\times \rho(\mathbf{B}), \quad N := \mathbb{C}^\times \rho(\mathbf{N}), \quad T := \mathbb{C}^\times \rho(\mathbf{T}).$$

We define $M(\rho)$ to be the monoid generated by G and certain linear projectors on V associated to the faces of the convex hull of the highest weight orbit $W\mu$. In particular,

G is the unit group of $M(\boldsymbol{\rho})$. Among others we obtain the following results, which show that $M(\boldsymbol{\rho})$ has similar properties to those of a \mathcal{J} -irreducible reductive linear algebraic monoid: We have a Bruhat decomposition

$$M(\boldsymbol{\rho}) = \bigsqcup_{x \in R} BxB \quad (\text{disjoint}),$$

and R is a generalized Renner monoid in the sense of E. Godelle [8, Definition 1.4], but we simply call it a Renner monoid. In particular, it is unit regular with unit group the Weyl group W , and its idempotent lattice $E(R)$ is isomorphic to the face lattice of the convex hull of the highest weight orbit $W\mu$. Furthermore, for $s \in S$ and $x \in R$ we have

$$sBx \subseteq BxB \cup BsxB \quad \text{and} \quad xBs \subseteq BxB \cup BxsB.$$

We have a decomposition

$$M(\boldsymbol{\rho}) = \bigsqcup_{e \in \Lambda} GeG \quad (\text{disjoint})$$

where Λ can be obtained by

$$\Lambda = \{e \in E(R) \mid Be = eBe\},$$

which is a complete sublattice of $E(R)$ and a cross-section for the action of W on $E(R)$ by conjugation. To every $e \in E(R)$ corresponds a unique idempotent of $M(\boldsymbol{\rho})$ which we denote by the same symbol. The set of idempotents of $M(\boldsymbol{\rho})$ is given by

$$E(M(\boldsymbol{\rho})) = \{geg^{-1} \mid e \in \Lambda, g \in G\}.$$

In particular, the monoid $M(\boldsymbol{\rho})$ is unit regular. Define the maps $\lambda^* : \Lambda \rightarrow 2^S$ and $\lambda_* : \Lambda \rightarrow 2^S$ on $e \in \Lambda$ by

$$\lambda^*(e) := \{s \in S \mid se = es \neq e\} \quad \text{and} \quad \lambda_*(e) := \{s \in S \mid se = es = e\}.$$

Define $\lambda : \Lambda \rightarrow 2^S$ on $e \in \Lambda$ by $\lambda(e) := \lambda^*(e) \cup \lambda_*(e)$. Then for $e, f \in \Lambda \setminus \{0\}$ we have

$$e \geq f \quad \text{if and only if} \quad \lambda^*(e) \supseteq \lambda^*(f).$$

Set $J_0 := \{s \in S \mid s\mu = \mu\}$. For $e \in \Lambda \setminus \{0\}$ the following are equivalent:

- (i) $I = \lambda^*(e)$ for some $e \in \Lambda \setminus \{0\}$.
- (ii) Every connected component of I intersects $S \setminus J_0$ nontrivially.

Furthermore, $\lambda_*(e) = \{s \in J_0 \setminus \lambda^*(e) \mid st = ts \text{ for all } t \in \lambda^*(e)\}$. Moreover, the left and right centralizers of $e \in \Lambda$ in G , which are the monoids

$$P(e) := \{g \in G \mid ge = ege\} \quad \text{and} \quad P^-(e) := \{g \in G \mid eg = ege\},$$

coincide with the opposite parabolic subgroups $P_{\lambda(e)}$ and $P_{\lambda(e)}^-$ of G .

All these results are reached purely algebraically by explicit calculations for which several sorts of stabilizers have to be determined. In particular, this requires the investigation of the weight strings through weights contained in the faces of the convex hull of the highest weight orbit $W\mu$.

Another class of analogues of reductive algebraic monoids whose unit groups are Kac-Moody groups, called face monoids associated to Kac-Moody groups, have been described and investigated in [26]-[31]. Here the idempotent lattice of a Renner monoid is isomorphic to the face lattice of the Tits cone. It is surprising and unexpected that such analogues exist, because they reduce classically to the groups themselves. In general, the face lattice of the Tits cone and the face lattice of the convex hull of a Weyl group orbit in the Tits cone differ drastically. In particular, the face monoids have one idempotent different from the unit for affine and strongly hyperbolic Kac-Moody groups and infinitely many idempotents for indefinite, not strongly hyperbolic Kac-Moody groups, whereas the monoids discussed in this article have infinitely many idempotents for affine and indefinite Kac-Moody groups.

In some subsequent articles the authors investigate the analogues of normal reductive algebraic monoids over \mathbb{C} whose unit groups are general Kac-Moody groups (Kac-Moody groups which generalize reductive linear algebraic groups). A first step of this program has been reached in [32] by describing the faces and face lattices of arbitrary Coxeter group invariant convex subcones of the Tits cone for a certain class of root bases, where the simple roots and simple coroots may be linearly dependent.

The contents of the sections of this article are the following: In Section 2 we collect some basic facts about Kac-Moody algebras, Kac-Moody groups, their images under highest weight representations, and some needed facts from convex geometry. It is unfortunate that sometimes the usual notation of Kac-Moody theory and of the theory of \mathcal{J} -irreducible reductive linear algebraic monoids are in conflict. In these instances we keep the notation of the latter. In Section 3 we establish some facts about the faces and the face lattice of the convex hull of a single Weyl group orbit in the Tits cone. Section 4 is the main part of the article. Here the monoid $M(\rho)$ is introduced and investigated. In particular, all the results stated above on $M(\rho)$ are proved.

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2 Preliminaries

We gather necessary notation and some basic facts about Kac-Moody algebras, Kac-Moody groups, and convex geometry from [10, 11, 13, 15, 27, 33, 38].

We denote by $\mathbb{Z}_{>}$, $\mathbb{R}_{>}$ the sets of strictly positive numbers of \mathbb{Z} , \mathbb{R} , and \mathbb{Z}_+ , \mathbb{R}_+ contain in addition the zero. If $M = \bigcup_{i \in I} M_i$ is a disjoint union of sets we write $M = \bigsqcup_{i \in I} M_i$ briefly.

2.1 Kac-Moody algebras

A *generalized Cartan matrix* is an integral matrix $A = (a_{ij})_{i,j=1}^m$ such that $a_{ii} = 2$, $a_{ij} \leq 0$ for $i \neq j$, and $a_{ij} = 0$ implies $a_{ji} = 0$. In this article we fix such a matrix A of rank l .

A realization of A is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$, where \mathfrak{h} is a $(2m - l)$ -dimensional complex vector space, $\Pi = \{\alpha_1, \dots, \alpha_m\} \subset \mathfrak{h}^*$ and $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_m^\vee\} \subset \mathfrak{h}$ are linearly independent subsets such that $\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij}$, where $\langle \cdot, \cdot \rangle : \mathfrak{h}^* \times \mathfrak{h} \rightarrow \mathbb{C}$ is the natural pairing and $i, j \in \mathbf{m} = \{1, \dots, m\}$. There exists a realization of A , unique up to isomorphism.

Let $\tilde{\mathfrak{g}}(A)$ be the complex Lie algebra generated by the abelian Lie algebra \mathfrak{h} and the symbols e_i, f_i , where $i \in \mathbf{m}$, with the following relations

$$[e_i, f_j] = \delta_{ij} \alpha_i^\vee, \quad [h, e_i] = \langle \alpha_i, h \rangle e_i, \quad [h, f_i] = -\langle \alpha_i, h \rangle f_i,$$

where $i, j \in \mathbf{m}$, $h \in \mathfrak{h}$. There exists a biggest ideal of $\tilde{\mathfrak{g}}(A)$ which intersects \mathfrak{h} trivially. The *Kac-Moody algebra* $\mathfrak{g} = \mathfrak{g}(A)$ is the corresponding quotient Lie algebra. We keep the same notation for the images of e_i, f_i, \mathfrak{h} in \mathfrak{g} .

The set Π is called the *root basis* and Π^\vee the *coroot basis*; elements in Π are referred to as *simple roots* and those in Π^\vee *simple coroots*. The free abelian group

$$Q = \bigoplus_{i=1}^m \mathbb{Z} \alpha_i \subseteq \mathfrak{h}^*$$

is called the *root lattice*. The Lie algebra \mathfrak{g} has the *root space decomposition*

$$\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha \quad \text{where} \quad \mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \langle \alpha, h \rangle x \text{ for all } h \in \mathfrak{h}\},$$

and \mathfrak{g}_α is called the *root space* associated to α . In particular, $\mathfrak{g}_0 = \mathfrak{h}$, $\mathfrak{g}_{\alpha_i} = \mathbb{C} e_i$, $\mathfrak{g}_{-\alpha_i} = \mathbb{C} f_i$, $i \in \mathbf{m}$. The *set of roots* is

$$\Delta := \{\alpha \in Q \setminus \{0\} \mid \mathfrak{g}_\alpha \neq \{0\}\}.$$

The *Chevalley anti-involution* \star of \mathfrak{g} is determined by

$$(e_i)^\star = f_i, \quad (f_i)^\star = e_i, \quad h^\star = h, \quad \text{for all } i \in \mathbf{m}, \quad h \in \mathfrak{h}.$$

It satisfies $(\mathfrak{g}_\alpha)^\star = \mathfrak{g}_{-\alpha}$, $\alpha \in Q$.

We put $Q_+ := \sum_{i=1}^m \mathbb{Z}_+ \alpha_i$, $Q_- := -Q_+$, and give \mathfrak{h}^\star the partial order

$$\mu \geq \mu_1 :\Leftrightarrow \mu - \mu_1 \in Q_+.$$

Then $\Delta = \Delta_+ \cup \Delta_-$ where $\Delta_+ = \{\alpha \in \Delta \mid \alpha > 0\}$ is the set of positive roots and $\Delta_- = \{\alpha \in \Delta \mid \alpha < 0\}$ the set of negative roots. Accordingly, there is the triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \quad \text{with} \quad \mathfrak{n}_\pm := \bigoplus_{\alpha \in \Delta^\pm} \mathfrak{g}_\alpha,$$

and $(\mathfrak{n}_-)^\star = \mathfrak{n}_+$, $\mathfrak{h}^\star = \mathfrak{h}$, and $(\mathfrak{n}_+)^\star = \mathfrak{n}_-$.

As in [10, Section 4.7] we associate to a generalized Cartan matrix A its Dynkin diagram, a certain graph whose vertices can be identified with the elements of \mathbf{m} or Π . The connected components of the Dynkin diagram correspond to the indecomposable generalized Cartan submatrices of A . We call $I, J \subseteq \Pi$ separated if $\langle \alpha, \beta^\vee \rangle = 0$ for all $\alpha \in I$ and $\beta \in J$.

For the classification of the Kac-Moody algebras $\mathfrak{g}(A)$ whose generalized Cartan matrices A are indecomposable into finite, affine, and indefinite type we refer to Chapter 4 of [10]. The Kac-Moody algebra $\mathfrak{g}(A)$ is of strongly hyperbolic type if it is of indefinite type and every proper nonempty indecomposable generalized Cartan submatrix of A is of finite type.

For each $i \in \mathbf{m}$ define the *fundamental reflection* $r_i \in GL(\mathfrak{h}^\star)$ by

$$r_i(\mu) = \mu - \langle \mu, \alpha_i^\vee \rangle \alpha_i, \quad \text{for all } \mu \in \mathfrak{h}^\star.$$

The Weyl group W of \mathfrak{g} is the subgroup of $GL(\mathfrak{h}^\star)$ generated by $S := \{r_i \mid i \in \mathbf{m}\}$. Moreover, (W, S) is a Coxeter system.

For $I, J \subseteq \Pi$ we denote by W_I the standard parabolic subgroup generated by I . We denote by ${}^I W$ the set of minimal coset representatives of $W_I \backslash W$, and by W^J the set of minimal coset representatives of W/W_J . We use ${}^I W^J$ to denote the set of minimal coset representatives of $W_I \backslash W/W_J$.

A *real root* is an element of $\Delta^{re} = W\Pi$, and an *imaginary root* is an element of $\Delta^{im} = \Delta \setminus \Delta^{re}$. If $\alpha \in \Delta^{re}$, then $\dim \mathfrak{g}_\alpha = 1$ and $\Delta \cap \mathbb{Z}\alpha = \{\alpha, -\alpha\}$. If $\alpha \in \Delta^{im}$, then $\mathbb{Z}\alpha \subset \Delta \cup \{0\}$.

The Weyl group W acts dually on \mathfrak{h} . In particular, for $i \in \mathbf{m}$ we have

$$r_i(h) = h - \langle \alpha_i, h \rangle \alpha_i, \quad \text{for all } h \in \mathfrak{h}.$$

A *real coroot* is an element of $(\Delta^{re})^\vee = W\Pi^\vee$. We obtain a W -equivariant bijective map ${}^\vee : \Delta^{re} \rightarrow (\Delta^{re})^\vee$ by mapping $\alpha = w\alpha_i$ to $\alpha^\vee = w\alpha_i^\vee$, $w \in W$, $i \in \mathbf{m}$. Furthermore, $\alpha > 0$ if and only if $\alpha^\vee > 0$, the partial order on \mathfrak{h} defined similarly as on \mathfrak{h}^\star .

The reflection with respect to $\alpha \in \Delta^{re}$ is defined by

$$r_\alpha(\mu) = \mu - \langle \mu, \alpha^\vee \rangle \alpha, \quad \text{for all } \mu \in \mathfrak{h}^*.$$

If $\alpha = w(\alpha_i)$ for some $w \in W$ and $i \in \mathbf{m}$, then $r_\alpha = wr_iw^{-1}$. In particular, $r_{\alpha_i} = r_i$.

Let $\mathfrak{h}_\mathbb{R} \subset \mathfrak{h}$ be a real vector space of dimension $2m - l$ such that $(\mathfrak{h}_\mathbb{R}, \Pi, \Pi^\vee)$ is a realization of A over \mathbb{R} . Then $\mathfrak{h}_\mathbb{R}^* \subset \mathfrak{h}^*$ is stable under W . The set

$$\overline{C} = \{\mu \in \mathfrak{h}_\mathbb{R}^* \mid \langle \mu, \alpha^\vee \rangle \geq 0, \text{ for all } \alpha \in \Pi\}$$

is called the fundamental chamber, and

$$X = \bigcup_{w \in W} w\overline{C}$$

is referred to as the Tits cone.

For $J \subseteq \Pi$ we set

$$\begin{aligned} C_J &= \{\mu \in \mathfrak{h}_\mathbb{R}^* \mid \langle \mu, \alpha^\vee \rangle = 0 \text{ for all } \alpha \in J, \langle \mu, \alpha^\vee \rangle > 0 \text{ for all } \alpha \in \Pi \setminus J\}, \\ \overline{C}_J &= \{\mu \in \mathfrak{h}_\mathbb{R}^* \mid \langle \mu, \alpha^\vee \rangle = 0 \text{ for all } \alpha \in J, \langle \mu, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Pi \setminus J\}. \end{aligned}$$

We call C_J the open, and \overline{C}_J the closed standard facet of type J . In particular, $\overline{C}_\emptyset = \overline{C}$. For every $w \in W$ we call wC_J an open, and $w\overline{C}_J$ a closed facet of type J .

Any W -orbit contained in X intersects the fundamental chamber \overline{C} at exactly one point. We have

$$\overline{C} = \bigsqcup_{J \subseteq \Pi} C_J,$$

and for every $\mu \in C_J$ its isotropy group $W_\mu := \{w \in W \mid w\mu = \mu\}$ is given by the standard parabolic subgroup W_J . We also call J the type of μ .

We have $wC_J = w'C_{J'}$ if and only if $J = J'$ and $wW_J = w'W_{J'}$. The open facets $\{wC_J \mid w \in W, J \subseteq \Pi\}$ give a W -invariant partition of the Tits cone X .

A \mathfrak{g} -module V is called \mathfrak{h} -diagonalizable if

$$V = \bigoplus_{\eta \in \mathfrak{h}^*} V_\eta \quad \text{where} \quad V_\eta = \{v \in V \mid hv = \langle \eta, h \rangle v \text{ for all } h \in \mathfrak{h}\},$$

and V_η is called the weight space associated to η . The set of weights is

$$P(V) := \{\eta \in \mathfrak{h}^* \mid V_\eta \neq \{0\}\}.$$

An \mathfrak{h} -diagonalizable \mathfrak{g} -module V is called *integrable* if e_i and f_i are locally nilpotent on V for all $i \in \mathbf{m}$. Its set of weights $P(V)$ is W -invariant. For $\eta \in P(V)$ and $\alpha \in \Delta^{re}$ the set $P(V) \cap (\eta + \mathbb{Z}\alpha)$ is called the α -weight string through η . If V_η is finite dimensional then it is of the form

$$\eta - p\alpha, \dots, \eta - \alpha, \eta, \eta + \alpha, \dots, \eta + q\alpha,$$

where p and q are nonnegative integers and $p - q = \langle \eta, \alpha^\vee \rangle$.

Associated to each $\mu \in \mathfrak{h}^*$ is, up to isomorphism, a unique irreducible highest weight module V with highest weight μ . It is \mathfrak{h} -diagonalizable with finite dimensional weight spaces. There is a nondegenerate symmetric contravariant bilinear form on V , unique up to a non-zero multiplicative scalar, such that

$$(gx | y) = (x | g^* y) \quad \text{for } g \in \mathfrak{g}, x, y \in V.$$

Moreover, V is integrable if and only if $\langle \mu, \alpha_i^\vee \rangle \in \mathbb{Z}^+$ for all $i \in \mathbf{m}$.

2.2 Kac-Moody groups

There are different versions of Kac-Moody groups. We use the one given in [27, 33], which will be described below. Others can be found in [11, 13, 15, 41, 42, 43]. The construction requires certain dual free abelian groups as additional data, which are used to specify a torus algebraic geometrically.

The set Π^\vee can be extended to a basis of \mathfrak{h} by adding elements $\alpha_{m+1}^\vee, \dots, \alpha_{2m-l}^\vee \in \mathfrak{h}_\mathbb{R}$ such that $\langle \alpha_j, \alpha_i^\vee \rangle \in \mathbb{Z}$ for $j \in \mathbf{m}$ and $i = m+1, \dots, 2m-l$. Let

$$\mathfrak{h}_\mathbb{Z} := \mathbb{Z}\alpha_1^\vee + \dots + \mathbb{Z}\alpha_{2m-l}^\vee \subset \mathfrak{h}_\mathbb{R} \subset \mathfrak{h},$$

which is a free abelian group of rank $2m-l$. It is W -invariant.

Let $\{\mu_1, \dots, \mu_{2m-l}\}$ be the basis of \mathfrak{h}^* dual to the basis $\{\alpha_1^\vee, \dots, \alpha_{2m-l}^\vee\}$ of \mathfrak{h} . The *weight lattice*

$$P := \mathbb{Z}\mu_1 + \dots + \mathbb{Z}\mu_{2m-l} \subset \mathfrak{h}_\mathbb{R}^* \subset \mathfrak{h}^*$$

is a free abelian group of rank $2m-l$, which is dual to $\mathfrak{h}_\mathbb{Z}$. It is W -invariant. Note also that $\Pi \subset P$, and hence $Q \subseteq P$. The elements of P are called *integral weights*, or simply *weights*. The elements of

$$\mu \in P_+ := P \cap \overline{C} = \{\mu \in P \mid \langle \mu, \alpha_i^\vee \rangle \in \mathbb{Z}_+ \text{ for all } i \in \mathbf{m}\}$$

are called *dominant integral weights*, or simply *dominant weights*. The elements $\mu_1, \dots, \mu_m \in P^+$ are called *fundamental dominant weights*.

An integrable representation (V, ρ) of \mathfrak{g} is called *admissible* if $P(V) \subseteq P$. The adjoint representation of \mathfrak{g} is admissible. Let $\widetilde{\mathbf{G}}$ be the free product of the torus

$$\text{Hom}((P, +), (\mathbb{C}^\times, \cdot))$$

and the additive groups \mathfrak{g}_α for $\alpha \in \Delta^{re}$. For any admissible representation (V, ρ) of \mathfrak{g} , there is a unique group homomorphism $\tilde{\rho} : \widetilde{\mathbf{G}} \rightarrow GL(V)$ which maps $\chi \in \text{Hom}(P, \mathbb{C}^\times)$ to $\chi_\rho \in GL(V)$ given by

$$\chi_\rho(v_\eta) := \chi(\eta)v_\eta \quad \text{if } v_\eta \in V_\eta, \eta \in P(V),$$

and which maps $x_\alpha \in \mathfrak{g}_\alpha$ to $\exp(\rho(x_\alpha)) \in GL(V)$, $\alpha \in \Delta^{re}$.

Let \widetilde{K} be the intersection of the kernels of all homomorphisms $\tilde{\rho}'$ where the intersection is taken over all admissible representations (V', ρ') . The Kac-Moody group is defined to be

$$\mathbf{G} = \widetilde{\mathbf{G}} / \widetilde{K}.$$

Let $q : \widetilde{\mathbf{G}} \rightarrow \mathbf{G}$ be the natural quotient homomorphism. Then there is a unique group representation $\rho : \mathbf{G} \rightarrow GL(V)$ such that the diagram

$$\begin{array}{ccc} \widetilde{\mathbf{G}} & \xrightarrow{\widetilde{\rho}} & GL(V) \\ q \downarrow & \nearrow \rho & \\ \mathbf{G} & & \end{array}$$

commutes.

Let $i_0 : \text{Hom}(P, \mathbb{C}^\times) \rightarrow \widetilde{\mathbf{G}}$ and $i_\alpha : \mathfrak{g}_\alpha \rightarrow \widetilde{\mathbf{G}}$, $\alpha \in \Delta^{re}$, be the canonical inclusions. The composition $\mathbf{t} := qi_0 : \text{Hom}(P, \mathbb{C}^\times) \rightarrow \mathbf{G}$ is an injective group homomorphism. Its image \mathbf{T} is called the torus of \mathbf{G} associated to \mathfrak{h} . For any $\alpha \in \Delta^{re}$ the composition $\exp := qi_\alpha : \mathfrak{g}_\alpha \rightarrow \mathbf{G}$ is an injective group homomorphism. Its image \mathbf{U}_α is called the root group of \mathbf{G} associated to \mathfrak{g}_α or simply to α . Note that

$$\begin{aligned} \rho(\mathbf{t}(\chi)) &= \chi_\rho & \text{for all } \chi \in \text{Hom}(P, \mathbb{C}^\times), \\ \rho(\exp(x)) &= \exp(\rho(x)) & \text{for all } x \in \mathfrak{g}_\alpha, \alpha \in \Delta^{re}, \end{aligned}$$

for every admissible representation (V, ρ) of \mathfrak{g} .

The torus \mathbf{T} can be also described in a different way. For every $h \in \mathfrak{h}_\mathbb{Z}$ and $s \in \mathbb{C}^\times$ we get an element $\chi_h(s) \in \text{Hom}(P, \mathbb{C}^\times)$ by $\chi_h(s)\eta := s^{\langle \eta, h \rangle}$, $\eta \in P$, inducing an isomorphism of the abelian groups $(\mathfrak{h}_\mathbb{Z}, +) \otimes_\mathbb{Z} (\mathbb{C}^\times, \cdot)$ and $\text{Hom}((P, +), (\mathbb{C}^\times, \cdot))$. For every $h \in \mathfrak{h}_\mathbb{Z}$ and $s \in \mathbb{C}^\times$ the corresponding element $t_h(s) := \mathbf{t}(\chi_h(s)) \in \mathbf{T}$ acts on each admissible representation (V, ρ) of \mathfrak{g} by

$$\rho(t_h(s))v_\eta = s^{\langle \eta, h \rangle}v_\eta, \quad v_\eta \in V_\eta, \quad \eta \in P(V).$$

Each element $t \in \mathbf{T}$ can be written uniquely as

$$t = \prod_{i=1}^{2m-l} t_i(c_i), \quad \text{for some } c_1, \dots, c_{2m-l} \in \mathbb{C}^\times,$$

where we have set $t_i(c_i) := t_{h_i}(c_i)$, $i = 1, \dots, 2m-l$.

For every $\alpha \in \Delta^{re}$ we choose $e_\alpha \in \mathfrak{g}_\alpha$, $f_\alpha \in \mathfrak{g}_{-\alpha}$ such that $[e_\alpha, f_\alpha] = \alpha^\vee$. For simplicity we choose $e_{\alpha_i} = e_i$ and $f_{\alpha_i} = f_i$ for all $i \in \mathbf{m}$. We choose $e_{-\alpha} = f_\alpha$ and $f_{-\alpha} = e_\alpha$ for all $\alpha \in \Delta_+^{re}$. There is a unique injective homomorphism $\varphi_\alpha : \text{SL}_2(\mathbb{C}) \rightarrow \mathbf{G}$ satisfying

$$\varphi_\alpha \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} = \exp(cx_\alpha) \quad \text{and} \quad \varphi_\alpha \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = \exp(cx_{-\alpha}), \quad \text{for all } c \in \mathbb{C}.$$

Its image $\mathbf{SL}_2^{(\alpha)}$ is the subgroup of \mathbf{G} generated by \mathbf{U}_α and $\mathbf{U}_{-\alpha}$. It induces an injective homomorphism of groups $n_\alpha : \mathbb{C}^\times \rightarrow \mathbf{SL}_2^{(\alpha)}$ by

$$n_\alpha(c) := \varphi_\alpha \begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix}, \quad c \in \mathbb{C}^\times,$$

and we have

$$n_\alpha(c) = \exp(ce_\alpha) \exp(-c^{-1}f_\alpha) \exp(ce_\alpha) = \exp(-c^{-1}f_\alpha) \exp(ce_\alpha) \exp(-c^{-1}f_\alpha)$$

and

$$t_{\alpha^\vee}(c) = n_\alpha(c)n_\alpha(1)^{-1} = \varphi_\alpha \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}, \quad c \in \mathbb{C}^\times.$$

We write $n_i(c)$ for $n_{\alpha_i}(c)$ and we write n_i for $n_{\alpha_i}(1)$, $i = 1, \dots, m$. We denote by \mathbf{N} the subgroup of \mathbf{G} generated by the elements of the torus \mathbf{T} and $n_\alpha(1)$, $\alpha \in \Delta^{re}$.

The system $(\mathbf{G}, \mathbf{T}, (\mathbf{U}_\alpha)_{\alpha \in \Delta^{re}})$ is a root group data system with associated twin BN-pairs $(\mathbf{B}^\pm, \mathbf{N})$. These structures are explained in [1, Chapter 8]. In particular,

$$\bigcap_{\alpha \in \Delta^{re}} N_{\mathbf{G}}(\mathbf{U}_\alpha) = \mathbf{T} \quad \text{and} \quad \mathbf{B}^+ \cap \mathbf{B}^- = \mathbf{B}^+ \cap \mathbf{N} = \mathbf{B}^- \cap \mathbf{N} = \mathbf{T}$$

where $N_{\mathbf{G}}(\mathbf{U}_\alpha)$ denotes the normalizer of \mathbf{U}_α in \mathbf{G} .

The common Weyl group \mathbf{N}/\mathbf{T} contains the set of simple reflections $\mathbf{S} = \{n_i \mathbf{T} \mid i \in \mathbf{m}\}$. As a Coxeter system it is isomorphic to the Weyl group W of the Kac-Moody Lie algebra \mathfrak{g} with its set of simple reflections $\{r_i \mid i \in \mathbf{m}\}$. We identify these Coxeter systems. If $w \in W$ is represented by $n_w \in \mathbf{N}$ then for every $\alpha \in \Delta^{re}$ we have $n_w \mathbf{U}_\alpha n_w^{-1} = \mathbf{U}_{w\alpha}$. For every admissible \mathfrak{g} -module V we have

$$n_w V_\eta = V_{w\eta} \quad \text{for all} \quad \eta \in P(V). \quad (2)$$

The Kac-Moody group \mathbf{G} has the Birkhoff and Bruhat decompositions,

$$\mathbf{G} = \bigsqcup_{w \in W} \mathbf{B}^- w \mathbf{B} = \bigsqcup_{w \in W} \mathbf{B} w \mathbf{B}^- \quad \text{and} \quad \mathbf{G} = \bigsqcup_{w \in W} \mathbf{B} w \mathbf{B} = \bigsqcup_{w \in W} \mathbf{B}^- w \mathbf{B}^-.$$

Let $I \subseteq \Pi$. The standard parabolic subgroups $\mathbf{P}_I = \mathbf{B} W_I \mathbf{B}$ and $\mathbf{P}_I^- = \mathbf{B}^- W_I \mathbf{B}^-$ admit the Levi decompositions

$$\mathbf{P}_I^\pm = \mathbf{L}_I \ltimes \mathbf{U}_\pm^I.$$

Here $\mathbf{L}_I = \mathbf{P}_I \cap \mathbf{P}_I^-$ is the subgroup of \mathbf{G} generated by \mathbf{T} and the root groups \mathbf{U}_α , $\alpha \in W_I I$. Furthermore, $\mathbf{U}_\pm^I = \cap_{w \in W_I} w \mathbf{U}^\pm w^{-1}$ are the normal subgroups of \mathbf{U}^\pm generated by the root groups \mathbf{U}_α , $\alpha \in \Delta_\pm^{re} \setminus W_I I$. In particular,

$$\mathbf{B}^\pm = \mathbf{T} \ltimes \mathbf{U}^\pm$$

where \mathbf{U}^\pm is the subgroup of \mathbf{G} generated by the root groups \mathbf{U}_α , $\alpha \in \Delta_\pm^{re}$.

The center of the Kac-Moody group \mathbf{G} is

$$Z(\mathbf{G}) = \left\{ \prod_{i=1}^{2m-l} t_i(c_i) \in \mathbf{T} \mid \prod_{i=1}^{2m-l} c_i^{\langle \alpha, \alpha_i^\vee \rangle} = 1 \text{ for all } \alpha \in \Delta^{re} \right\} \subseteq \mathbf{T}.$$

An irreducible highest weight representation (V, ρ) of \mathfrak{g} with highest weight μ is admissible if and only if $\mu \in P^+$. We call the corresponding representation (V, ρ) of \mathbf{G} an irreducible highest weight representation of \mathbf{G} of highest weight μ . The following Lemma lists its kernel in two cases. To prove (b), [2, Chapter IV, §2.7, Lemma 2] is useful.

Lemma 2.1 *Let (V, ρ) be an irreducible highest weight representation of \mathbf{G} of highest weight $\mu \in P^+$. Let J_0 be the type of μ .*

(a) *The following are equivalent:*

- (1) $\mathbf{G}' \subseteq \ker(\rho)$.
- (2) $J_0 = \Pi$.
- (3) $\dim V = 1$.

(b) *If no connected component of Π is contained in J_0 then*

$$\ker(\rho) = \left\{ \prod_{i=1}^{2m-l} t_i(c_i) \in Z(\mathbf{G}) \mid \prod_{i=1}^{2m-l} c_i^{\langle \mu, \alpha_i^\vee \rangle} = 1 \right\} \subseteq Z(\mathbf{G}) \subseteq \mathbf{T}.$$

The Chevalley anti-involution \star of \mathfrak{g} induces an anti-involution of the group \mathbf{G} , denoted still by \star , defined by

$$t^\star = t \quad \text{and} \quad (\exp(x_\alpha))^\star = \exp((x_\alpha)^\star) \quad \text{for } t \in \mathbf{T}, x_\alpha \in \mathfrak{g}_\alpha, \alpha \in \Delta^{re}.$$

If (V, ρ) is an irreducible highest weight representation of \mathbf{G} and $(|)$ a contravariant nondegenerate symmetric bilinear form on V then

$$(gx | y) = (x | g^\star y) \quad \text{for all } x, y \in V, g \in \mathbf{G}.$$

2.3 The group $\mathbb{C}^\times \rho(\mathbf{G})$

We now fix an irreducible highest weight representation (V, ρ) of \mathbf{G} of highest weight $\mu \in P^+$, such that no connected component of Π is contained in the type J_0 of μ . We fix a contravariant nondegenerate symmetric bilinear form $(|)$ on V .

Viewing each element $c \in \mathbb{C}^\times$ as the natural scalar multiplication by c on V , the product group

$$G := \mathbb{C}^\times \rho(\mathbf{G})$$

is a subgroup of $\text{End}(V)$, and its center contains \mathbb{C}^\times . This group is the unit group of the monoid $M(\rho)$ we will define in Section 4.

Because of Lemma 2.1 (b) the group G inherits most structures from \mathbf{G} . Define

$$T = \mathbb{C}^\times \rho(\mathbf{T}) \quad \text{and} \quad U_\alpha = \rho(U_\alpha), \alpha \in \Delta^{re}.$$

Then $(G, T, (U_\alpha)_{\alpha \in \Delta^{re}})$ is root group data system with associated twin BN-pairs

$$B^\pm := \mathbb{C}^\times \rho(B^\pm) \quad \text{and} \quad N := \mathbb{C}^\times \rho(N).$$

These structures are explained in [1, Chapter 8]. In particular, we have

$$\bigcap_{\alpha \in \Delta^{re}} N_G(U_\alpha) = T \quad \text{and} \quad B^+ \cap B^- = B^+ \cap N = B^- \cap N = T$$

where $N_G(U_\alpha)$ denotes the normalizer of U_α in G .

The common Weyl group N/T contains the set of simple reflections $S = \{\rho(n_i)T \mid i \in \mathbf{m}\}$. As a Coxeter system it is isomorphic to the Weyl group W of the Kac-Moody Lie algebra \mathfrak{g} with its set of simple reflections $\{r_i \mid i \in \mathbf{m}\}$. We identify these Coxeter systems. If $w \in W$ is represented by $n_w \in N$ then $n_w U_\alpha n_w^{-1} = U_{w\alpha}$ for all $\alpha \in \Delta^{re}$, and

$$n_w V_\eta = V_{w\eta} \quad \text{for all } \eta \in P(V). \quad (3)$$

The group G has the Birkhoff and Bruhat decompositions,

$$G = \bigsqcup_{w \in W} B^- w B = \bigsqcup_{w \in W} B w B^- \quad \text{and} \quad G = \bigsqcup_{w \in W} B w B = \bigsqcup_{w \in W} B^- w B^-.$$

It is often convenient to identify the subset $I \subseteq \Pi$ with the set of indices of the simple roots in I . We will do so as needed. The standard parabolic subgroups $P_I = B W_I B$ and $P_I^- = B^- W_I B^-$ admit the Levi decompositions

$$P_I^\pm = L_I \ltimes U_\pm^I.$$

where the standard Levi subgroup $L_I = P_I \cap P_I^-$ is the subgroup of G generated by T and the root groups U_α , $\alpha \in W_I I$. Furthermore, $U_\pm^I = \cap_{w \in W_I} w U^\pm w^{-1}$, which are the normal subgroups of U^\pm generated by the root groups U_α , $\alpha \in \Delta_\pm^{re} \setminus W_I I$. In particular,

$$B^\pm = T \ltimes U^\pm$$

where U^\pm is the subgroup of G generated by the root groups U_α , $\alpha \in \Delta_\pm^{re}$, and coincides with $\rho(U^\pm)$.

Let T_i be the subgroup of T generated by $\{\rho(t_i(c)) \mid c \in \mathbb{C}^\times\}$ for $i = 1, \dots, 2m-l$. Let T_I be the subgroup of T generated by T_i , $i \in I$, and put T^I to be the subgroup of T generated by T_i , $i \in \{1, \dots, 2m-l\} \setminus I$. Then

$$T = \mathbb{C}^\times (T_I T^I).$$

The standard Levi subgroup L_I and its derived group $G_I = L_I'$ have the Birkhoff and Bruhat decompositions

$$L_I = U_I^\pm T N_I U_I^\pm \quad \text{and} \quad G_I = U_I^\pm N_I U_I^\pm,$$

where $U_I^\pm = L_I \cap U^\pm = G_I \cap U^\pm$ is the subgroup of U^\pm generated by $U_{\pm\alpha}$, $\alpha \in I$, and $N_I = G_I \cap N$ is the subgroup of N generated by T_I and $\rho(n_i)$ for $i \in I$. Note also that G_I is generated by the root groups $U_{\pm\alpha}$, $\alpha \in I$, and $L_I = T G_I = \mathbb{C}^\times T^I G_I$.

The Chevalley anti-involution \star of \mathbf{G} induces an anti-involution of the group G , denoted still by \star , defined by

$$(c\rho(g))^\star := c\rho(g^\star), \quad c \in \mathbb{C}^\times, g \in G.$$

In particular, the adjoint of $g \in G$ with respect to the contravariant nondegenerate symmetric bilinear form $(|)$ on V exists and is given by g^\star .

2.4 Convex geometry

We collect some basic results about the facial structure of convex sets. These results will be used to describe the geometry of the convex hull of certain Weyl group orbits in $\mathfrak{h}_{\mathbb{R}}^*$. The main references for this part are [5, 38].

Lemma 2.2 ([38, Corollaries 18.1.2 and 18.1.3]) *Let H be a convex set in $\mathfrak{h}_{\mathbb{R}}^*$.*

(a) *If the intersection of the relative interiors of the faces F and F' of H is not empty, then $F = F'$.*

(b) *If F is a proper face of H , then F lies entirely in the relative boundary of H , and $\dim F < \dim H$.*

Lemma 2.3 ([5, Theorem 5.2]; [38, Section 18]) *Let F be a face of a convex set H , and let F' be a subset of F . Then F' is a face of H if and only if F' is a face of F .*

Lemma 2.4 ([38, Theorem 6.6]) *Let σ be a linear transformation from $\mathfrak{h}_{\mathbb{R}}^*$ to $\mathfrak{h}_{\mathbb{R}}^*$. If H is a convex set in $\mathfrak{h}_{\mathbb{R}}^*$, then*

$$\text{ri}(\sigma H) = \sigma(\text{ri } H) \quad \text{and} \quad \text{rb}(\sigma H) \supseteq \sigma(\text{rb } H).$$

Let $\mathcal{F}(H)$ be the set of all faces of a convex set H . We give $\mathcal{F}(H)$ the partial order

$$F \leq F', \quad \text{if } F \subseteq F'.$$

We agree that any subset of $\mathcal{F}(H)$ inherits this order. The set $\mathcal{F}(H)$ is a complete lattice, called the *face lattice* of H , where the meet is the intersection, and the join of two faces is the smallest face containing both faces.

Lemma 2.5 ([38, Theorem 18.2]) *If H is a nonempty convex set in $\mathfrak{h}_{\mathbb{R}}^*$, then the set*

$$\{\text{ri } F \mid F \in \mathcal{F}(H) \setminus \{\emptyset\}\}$$

is a partition of H .

Lemma 2.6 ([38, Theorem 18.3]) *Let F be a face of a convex set H . If H is generated by S as a convex set then F is generated by $S \cap F$ as a convex set.*

3 Geometry of W -Orbits

Let μ be in the fundamental chamber \overline{C} of the Tits cone X of a Kac-Moody algebra $\mathfrak{g}(A)$. We set

$$J_0 := \{\alpha \in \Pi \mid \langle \mu, \alpha^\vee \rangle = 0\} \quad \text{and} \quad J_{>} := \Pi \setminus J_0 = \{\alpha \in \Pi \mid \langle \mu, \alpha^\vee \rangle > 0\},$$

and call J_0 the type of μ . We call the convex hull

$$H := \text{co}(W\mu) \subseteq \mathfrak{h}_{\mathbb{R}}^*$$

of the Weyl group orbit $W\mu$ in $\mathfrak{h}_{\mathbb{R}}^*$ the *orbit hull* of μ . Clearly, H is W -invariant.

We denote by $\mathcal{F}(H)$ the *face lattice* of H . The action of W on H induces an action of W on $\mathcal{F}(H)$ by lattice isomorphisms.

Associated to each face $F \in \mathcal{F}(H)$ is its affine hull $\text{aff}(F)$, the smallest affine subspace containing F . Furthermore, its *isotropy group*

$$W(F) := \{w \in W \mid wF = F\}$$

in W , which is a subgroup of W , and its *stabilizer*

$$W_*(F) := \{w \in W \mid w\eta = \eta \text{ for all } \eta \in F\}$$

in W , which is a normal subgroup of $W(F)$. For $w \in W$ we have

$$\text{aff}(wF) = w \text{aff}(F) \quad \text{and} \quad W(wF) = wW(F)w^{-1} \quad \text{and} \quad W_*(wF) = wW_*(F)w^{-1}. \quad (4)$$

A face is called *fundamental* if it is either empty or if its relative interior intersects the fundamental chamber \overline{C} nontrivially. The vertex $\{\mu\}$ and the orbit hull H are fundamental faces. We set

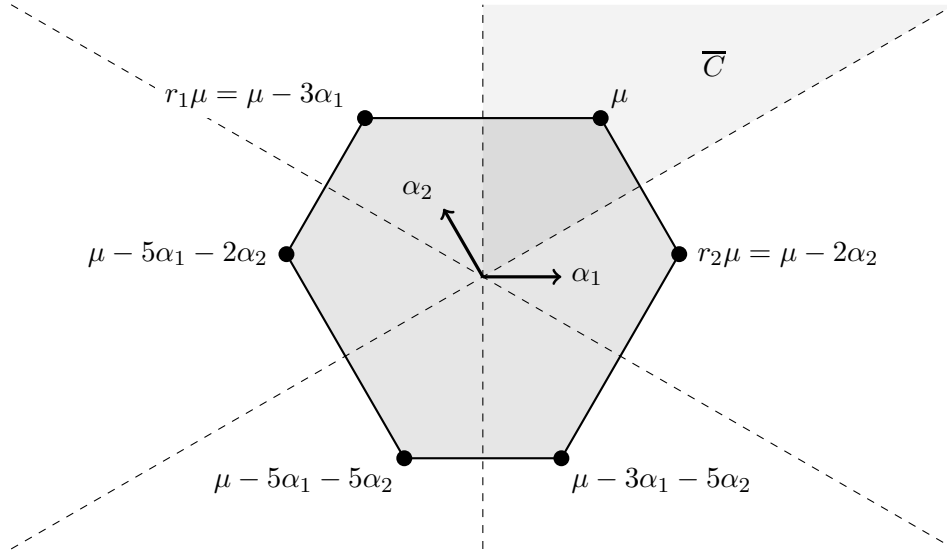
$$\mathcal{F} := \{F \in \mathcal{F}(H) \mid \overline{C} \cap \text{ri } F \neq \emptyset\} \cup \{\emptyset\}.$$

We give three examples which illustrate H and its faces for finite, affine, and strongly hyperbolic Kac-Moody algebras. For indefinite, not strongly hyperbolic Kac-Moody algebras the situation is more complicated as can be seen by Corollaries 3.13 and 3.19.

Example 3.1 Let $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Then the Kac-Moody algebra $\mathfrak{g}(A)$ is the simple Lie algebra of type A_2 . Its Tits cone is the whole space $\mathfrak{h}_{\mathbb{R}}^* = \mathbb{R}^2$. Let $\mu = 3\mu_1 + 2\mu_2 \in \overline{C}$. The Weyl group orbit

$$W\mu = \{\mu, \mu - 3\alpha_1, \mu - 5\alpha_1 - 2\alpha_2, \mu - 5\alpha_1 - 5\alpha_2, \mu - 3\alpha_1 - 5\alpha_2, \mu - 2\alpha_2\},$$

and its convex hull H are indicated in the following picture:



The Weyl group orbit $W\mu$ is contained on a circle. Its convex hull H has 14 faces: The empty face, 6 vertices, 6 edges, and H . The set of fundamental faces is

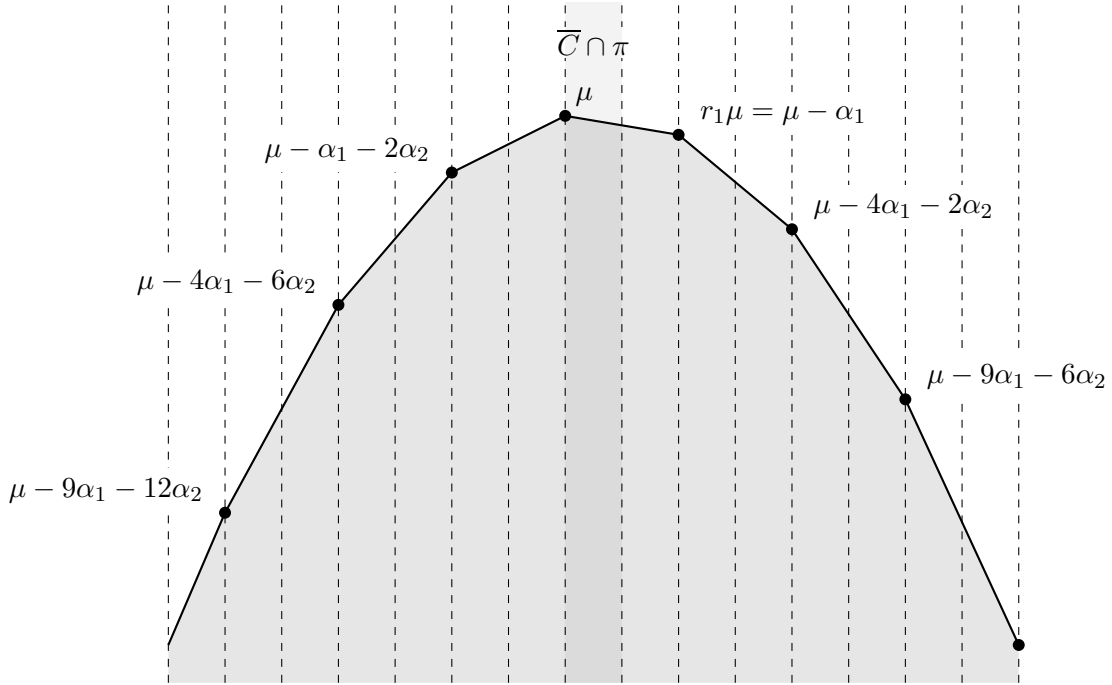
$$\mathcal{F} = \{\emptyset, \{\mu\}, \overline{\mu r_1 \mu}, \overline{\mu r_2 \mu}, H\},$$

where $\overline{\mu r_i \mu}$ is the closed line segment between μ and $r_i \mu$, $i = 1, 2$.

Example 3.2 Let $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. Then the Kac-Moody algebra $\mathfrak{g}(A)$ is the affine Lie algebra of type $A_1^{(1)}$. Its Tits cone in $\mathfrak{h}_{\mathbb{R}}^* = \mathbb{R}^3$ consists of the line $\mathbb{R}(\alpha_1 + \alpha_2)$ and an open half space bounded by $\mathbb{R}\alpha_1 + \mathbb{R}\alpha_2$. The Tits cone is subdivided by the reflection planes like an infinite open book by its pages. Let $\mu = \mu_1 \in \overline{C}$. The Weyl group orbit

$$W\mu = \{\mu - n^2\alpha_1 - n(n+1)\alpha_2 \mid n \in \mathbb{Z}\},$$

and its convex hull H are contained in the affine plane $\pi = \mu + \mathbb{R}\alpha_1 + \mathbb{R}\alpha_2$, which is contained in the Tits cone, [33, Appendix]. A part of this affine plane is indicated in the following picture.



The Weyl group orbit $W\mu$ is contained on a parabola. Its convex hull H has infinitely many faces. The set of fundamental faces is

$$\mathcal{F} = \{\emptyset, \{\mu\}, \overline{\mu r_1 \mu}, H\}.$$

Example 3.3 Let $A = \begin{pmatrix} 2 & -2 \\ -3 & 2 \end{pmatrix}$. Then $\mathfrak{g}(A)$ is a Kac-Moody algebra of indefinite, strongly hyperbolic type. The form of its Tits cone in $\mathfrak{h}_{\mathbb{R}}^* = \mathbb{R}^2$ is an open wedge with

the additional point $\{0\}$. Let $\mu = \mu_1 + \mu_2 \in \overline{C}$. For $n \in \mathbb{Z}$ set

$$g(n) = f(n)^2 + \sqrt{\frac{2}{3}} f(n)f(n-1) \quad \text{and} \quad h(n) = f(n)^2 + \sqrt{\frac{3}{2}} f(n)f(n+1)$$

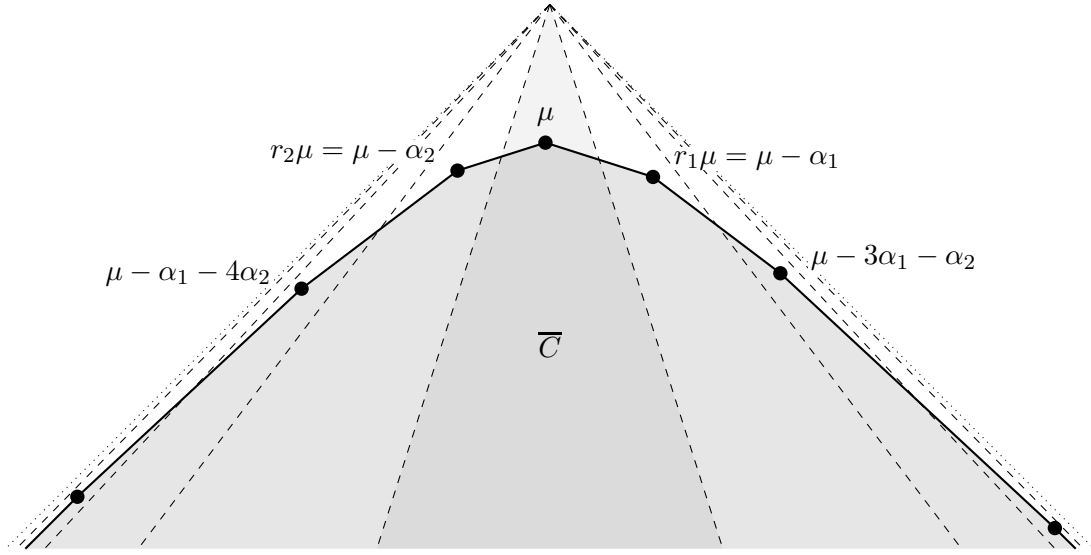
where

$$f(n) = \frac{\sinh(n\theta)}{\sinh(\theta)} \quad \text{with} \quad \theta = \text{Arcosh}\left(\frac{1}{2}\sqrt{2 \cdot 3}\right).$$

The Weyl group orbit

$$W\mu = \{\mu - g(n)\alpha_1 - h(n)\alpha_2, \mu - g(n+1)\alpha_1 - h(n)\alpha_2 \mid n \in \mathbb{Z}\},$$

and its convex hull H are indicated in the following picture:



The Weyl group orbit $W\mu$ is contained on a hyperbola. Its convex hull has infinitely many faces. The set of fundamental faces is

$$\mathcal{F} = \{\emptyset, \{\mu\}, \overline{\mu r_1 \mu}, \overline{\mu r_2 \mu}, H\}.$$

Classically, for finite Weyl groups, the face lattices of orbit hulls have been investigated by several people. A detailed literature survey can be found in [14, Section 2]. The description of \mathcal{F} in Corollary 3.14, the cross section lattice property of \mathcal{F} in Corollaries 3.17 and 3.22, and the descriptions of $W_*(F)$ and $W(F)$, $F \in \mathcal{F}$, in Theorem 3.15 have been obtained classically by W. A. Casselman in [4, Sections 3 and 4], building on some work of A. Borel, J. Tits in [3, Sections 12.14 - 12.17], and of I. Satake in [40, Section 2.3]. For orbit hulls of integral weights these results have been shown independently by E. B. Vinberg in [44, Section 3.1]. Equivalent versions have been obtained independently in the theory of \mathcal{J} -irreducible reductive linear algebraic monoids by M. S. Putcha and L. E. Renner in [35, Section 4].

The classical approaches for the investigation of the face lattice $\mathcal{F}(H)$ do not generalize to infinite Weyl groups. Instead we combine some ideas of [24, Section 4], [32, Section 4], and [30, Section 2]. In addition to the results mentioned above we obtain a description of the relative interiors of the faces of H , generalizing and completing a result of V. Kac and D. Peterson. We show that a criteria of E. Looijenga for the orbit hull H to be closed is actually a characterization. We reach a description of the lattice operations of the face lattice $\mathcal{F}(H)$, which seems to be new even classically.

For $I \subseteq \Pi$ we denote by

$$F_I := \text{co}(W_I \mu) \subseteq \mathfrak{h}_{\mathbb{R}}^*$$

the convex hull of $W_I \mu$ in $\mathfrak{h}_{\mathbb{R}}^*$. To prove that these sets are fundamental faces we use two Lemmas. The first can be found for example in [25, Proposition 1.11]:

Lemma 3.1 *We have $H \subset \mu - \mathbb{R}_+ \Pi$.*

The second Lemma generalizes [24, Lemma 4.2], as well as the fifth paragraph of [44, Section 3.1] for orbit hulls where the Weyl group is finite. M. Dyer obtains independently the same result by a different proof in [7, Lemma 2.4 (d)].

Lemma 3.2 *Let $w \in W$ and $I \subseteq \Pi$ with $w\mu \in \mu - \mathbb{R}_+ I$. Then there exists $u \in W_I$ such that $w\mu = u\mu$.*

Proof. We use induction on the length $l(w)$. It is clear that the result holds for $l(w) = 0$. Now suppose that $l(w) \geq 1$. By [1, Proposition 2.20] there exist $w^I \in W^I$, $w_I \in W_I$ such that $w = w^I w_I$ and $l(w) = l(w^I) + l(w_I)$. If $w^I = 1$ then $w \in W_I$, and the proof is complete. If $w^I \neq 1$, let $w^I = r_{\gamma_1} \dots r_{\gamma_k}$ be a reduced expression. Then $\gamma_k \notin I$. Moreover, $\alpha := w^I \gamma_k < 0$ and $w_I^{-1} \gamma_k > 0$ by [10, Lemma 3.11] and [1, Lemma 2.15]. Since $\alpha^\vee = w^I \gamma_k^\vee$ and $w_I^{-1} \gamma_k^\vee = (w_I^{-1} \gamma_k)^\vee > 0$ we obtain $\langle w\mu, \alpha^\vee \rangle = \langle w_I \mu, \gamma_k^\vee \rangle = \langle \mu, w_I^{-1} \gamma_k^\vee \rangle \geq 0$.

If $\langle w_I \mu, \gamma_k^\vee \rangle = 0$, then $r_{\gamma_k} w_I \mu = w_I \mu$. Hence, $w\mu = r_{\gamma_1} \dots r_{\gamma_{k-1}} w_I \mu$. From the induction hypothesis, there exists an element $u \in W_I$ such that $r_{\gamma_1} \dots r_{\gamma_{k-1}} w_I \mu = u\mu$. Thus, $w\mu = u\mu$.

If $\langle w_I \mu, \gamma_k^\vee \rangle = \langle w\mu, \alpha^\vee \rangle > 0$ we write $w\mu = \mu - \sum_{\beta \in I} a_\beta \beta$ with $a_\beta \in \mathbb{R}_+$. By Lemma 3.1 we have

$$r_\alpha w\mu = \mu - \sum_{\beta \in I} a_\beta \beta - \langle w\mu, \alpha^\vee \rangle \alpha \subseteq \mu - \mathbb{R}_+ \Pi,$$

since $r_\alpha w\mu \in H$. But α is a negative root, so it is a linear combination of simple roots from I . Thus $r_\alpha \in W_I$ and $r_\alpha w\mu \in \mu - \mathbb{R}_+ I$. Moreover, the length of $r_\alpha w = (w^I r_{\gamma_k}) w_I$ is smaller than the length of w . From the induction hypothesis, there exists $v \in W_I$ such that $r_\alpha w\mu = v\mu$. Hence $w\mu = r_\alpha v\mu$ with $r_\alpha v \in W_I$. \square

Theorem 3.3 *Let $I \subseteq \Pi$. Then F_I is an exposed fundamental face of H , and*

$$F_I = H \cap (\mu - \mathbb{R}_+ I) = W_I(F_I \cap \overline{C}).$$

Moreover, $W_I \mu = F_I \cap W\mu$.

Proof. As $\mu - \mathbb{R}_+ I$ is an exposed face of $\mu - \mathbb{R}_+ \Pi$ we obtain that $H \cap (\mu - \mathbb{R}_+ I)$ is an exposed face of $H \cap (\mu - \mathbb{R}_+ \Pi) = H$. Now Lemma 2.6 shows that $H \cap (\mu - \mathbb{R}_+ I)$ is generated by $W\mu \cap (\mu - \mathbb{R}_+ I)$ as a convex set, which coincides with $W_I \mu$ by Lemma 3.2. So $H \cap (\mu - \mathbb{R}_+ I) = F_I$.

Since F_I is the convex hull of $W_I \mu$ we have $W_I F_I = F_I$. Clearly, $W_I(F_I \cap \overline{C}) \subseteq F_I$. To show the reverse inclusion let $\eta_1 \in F_I$. Then there exists $w \in W$ and $\eta \in \overline{C}$ such that $\eta_1 = w\eta$. By Lemma 3.1 we have $\eta = w^{-1}\eta_1 \in H \subseteq \mu - \mathbb{R}_+ \Pi$. Applying Lemma 3.1 to η instead of μ we obtain

$$\eta_1 = w\eta = \eta - \sum_{\alpha \in \Pi} a_\alpha \alpha = \mu + (\eta - \mu) - \sum_{\alpha \in \Pi} a_\alpha \alpha = \mu - \sum_{\alpha \in \Pi} b_\alpha \alpha - \sum_{\alpha \in \Pi} a_\alpha \alpha$$

with $a_\alpha, b_\alpha \in \mathbb{R}_+$, $\alpha \in \Pi$. Since $\eta_1 \in F_I = H \cap (\mu - \mathbb{R}_+ I)$ we find that $a_\alpha = b_\alpha = 0$ for all $\alpha \in \Pi \setminus I$. Hence $\eta_1 = w\eta \in \eta - \mathbb{R}_+ I$. By Lemma 3.2, applied for η instead of μ , there exists $w_1 \in W_I$ such that $\eta_1 = w_1\eta$. We also get $\eta = w_1^{-1}\eta_1 \in W_I F_I = F_I$. Thus, $\eta \in F_I \cap \overline{C}$, and hence $\eta_1 \in W_I(F_I \cap \overline{C})$.

We next show that F_I is fundamental. Choose some $\gamma \in \text{ri}(F_I)$. Because of $F_I = W_I(F_I \cap \overline{C})$ there exists some $w \in W_I$ such that $w\gamma \in F_I \cap \overline{C}$. Moreover, $w\gamma \in w \text{ri}(F_I) = \text{ri}(wF_I) = \text{ri}(F_I)$. \square

A nonempty subset $I \subseteq \Pi$ is called μ -connected if every connected component of I intersects $J_>$ nontrivially. Note that I is μ -connected if and only if all its connected components are μ -connected. We agree that the empty set is μ -connected. The following proposition is obvious.

Proposition 3.4 *An arbitrary union of μ -connected sets is again μ -connected. For $I \subseteq \Pi$ there exists a biggest μ -connected set I^* contained in I , which we call the μ -connected part of I .*

Ordered partially by inclusion, the μ -connected sets form a lattice. The lattice meet and lattice join of two μ -connected sets I_1, I_2 are given by

$$I_1 \wedge I_2 = (I_1 \cap I_2)^* \quad \text{and} \quad I_1 \vee I_2 = I_1 \cup I_2.$$

Moreover, \emptyset is the smallest, and Π^ is the biggest μ -connected set.*

For $I \subseteq \Pi$ we set

$$I_* := \{\alpha \in J_0 \setminus I^* \mid r_\alpha r_\beta = r_\beta r_\alpha \text{ for all } \beta \in I^*\}. \quad (5)$$

Note also that for $\alpha, \beta \in \Pi$, $\alpha \neq \beta$, we have $r_\alpha r_\beta = r_\beta r_\alpha$ if and only if $\langle \beta, \alpha^\vee \rangle = 0$, if and only if $\langle \alpha, \beta^\vee \rangle = 0$.

The μ -connected part I^* is the union of the connected components of I which intersect $J_>$ nontrivially. Denote by I_r be the union of the connected components of I which intersect $J_>$ trivially. Then

$$I = I^* \sqcup I_r \quad \text{and} \quad I_r \subseteq I_*. \quad (6)$$

The following Proposition shows that for our investigations it is sufficient to consider the faces F_I for μ -connected sets I .

Proposition 3.5 *Let I be a subset of Π . Then $F_I = F_{I^*}$.*

Proof. We have $W_I\mu = W_{I^*}W_{I_r}\mu = W_{I^*}\mu$. Hence $F_I = F_{I^*}$. \square

Let I be a μ -connected set. Next we determine the affine hull of the face F_I . We obtain an interior point of F_I , whose isotropy group coincides, as we will see later, with the stabilizer $W_*(F_I)$ of the whole face F_I . Such points are useful for some proofs.

The way to do this is to construct a simplex of maximal dimension contained in F_I , which is formulated by the following technical lemma.

Lemma 3.6 *Let I be a nonempty μ -connected subset of Π .*

(a) *There exists a linear order on I with the following property: For every $\beta \in I$ there exists a chain $\gamma_0 < \gamma_1 < \dots < \gamma_l = \beta$ in I such that $\gamma_0 \in J_{>}$ and γ_{i-1}, γ_i are adjacent for $i = 1, \dots, l$.*

(b) *Let $I = \{\beta_1, \dots, \beta_k\}$ such that $\beta_1 < \beta_2 < \dots < \beta_k$ is a linear order as in (a). Then the convex hull S of*

$$\eta_0 := \mu, \quad \eta_1 := r_{\beta_1}\mu, \quad \eta_2 := r_{\beta_2}r_{\beta_1}\mu, \quad \dots \quad \eta_k := r_{\beta_k} \cdots r_{\beta_1}\mu$$

is a k -dimensional simplex, and its affine hull is $\mu + \mathbb{R}I$. Furthermore,

$$\text{ri}(S) \cap (\mu - \mathbb{R}_{>}I) \cap C_{I^*} \neq \emptyset. \quad (7)$$

Proof. To show (a) first assume that I is connected. For $\beta, \beta' \in I$ we define the distance $d(\beta, \beta')$ to be the minimum over the length l of all chains $\beta = \gamma_0, \gamma_1, \dots, \gamma_l = \beta'$ such that γ_{j-1}, γ_j are adjacent for $j = 1, \dots, l$. Since I is μ -connected there exists an element $\gamma_0 \in I \cap J_{>}$. Define $I(p) := \{\gamma \in I \mid d(\gamma_0, \gamma) = p\}$ for $p \in \mathbb{Z}_+$. Then $I(0) = \{\gamma_0\}$ and $I(p) = \emptyset$ at least for $p > |I|$. Ordering the elements in each of $I(0), I(1), I(2), \dots$ linearly and defining $I(0) < I(1) < I(2) < \dots$, we obtain a linear order on I with the property of (a).

Now let I have the connected components I_1, I_2, \dots, I_p with $p > 1$. Since these component are also μ -connected there exists on every component a linear order with the property of (a). We obtain a linear order on I with the property of (a) by defining $I_1 < I_2 < \dots < I_p$.

Next we prove the first part of (b). Set $b_t := \langle \eta_{t-1}, \beta_t^\vee \rangle$ for all $t = 1, \dots, k$. We first show that

$$\eta_t = \mu - b_1\beta_1 - b_2\beta_2 - \dots - b_t\beta_t \quad \text{and} \quad b_1, b_2, \dots, b_t \in \mathbb{R}_{>} \quad (8)$$

for all $t = 1, \dots, k$ by induction on t . If $t = 1$, then we have $r_{\beta_1}\mu = \mu - b_1\beta_1$ with $b_1 = \langle \mu, \beta_1^\vee \rangle > 0$ since $\beta_1 \in J_{>}$. Now let $2 \leq t \leq k$ and suppose that (8) holds for $t - 1$. Then

$$\eta_t = r_{\beta_t}\eta_{t-1} = \eta_{t-1} - \langle \eta_{t-1}, \beta_t^\vee \rangle \beta_t = \mu - b_1\beta_1 - \dots - b_{t-1}\beta_{t-1} - b_t\beta_t,$$

and

$$b_t = \langle \eta_{t-1}, \beta_t^\vee \rangle = \langle \mu - b_1\beta_1 - b_2\beta_2 - \dots - b_{t-1}\beta_{t-1}, \beta_t^\vee \rangle \geq 0$$

since $\langle \mu, \beta_t^\vee \rangle \geq 0$ and $-b_j \langle \beta_j, \beta_t^\vee \rangle \geq 0$ for all $j = 1, \dots, t-1$. If $\beta_t \in J_>$ then we have $\langle \mu, \beta_t^\vee \rangle > 0$. If $\beta_t \notin J_>$ then there exists $s < t$ such that β_s, β_t are adjacent, from which we get $-b_s \langle \beta_s, \beta_t^\vee \rangle > 0$. Thus $b_t > 0$.

Denote by $\text{lin}(S)$ the translation space of the affine hull $\text{aff}(S)$ of S . From (8) we obtain $b_t \beta_t = \eta_{t-1} - \eta_t \in \text{lin}(S) \subseteq \mathbb{R}I$ and $b_t \neq 0$ for all $t = 1, \dots, k$. It follows that $\text{lin}(S) = \mathbb{R}I$, and $\text{aff}(S) = \mu + \mathbb{R}I$. In particular, S is a k -dimensional simplex.

We now prove the second part of (b) by showing that there exist $\epsilon_1, \dots, \epsilon_k \in \mathbb{R}_>$ such that $x_t := \eta_0 + \epsilon_1 \eta_1 + \dots + \epsilon_t \eta_t$ is contained in

$$\{\eta \in (1 + \epsilon_1 + \dots + \epsilon_t)\mu - \mathbb{R}_>\{\beta_1, \dots, \beta_t\} \mid \langle \eta, \beta_j^\vee \rangle > 0 \text{ for } j = 1, \dots, t\} \quad (9)$$

for $t = 1, \dots, k$. We use induction on t . Let $t = 1$. Since $\beta_1 \in J_>$ we have $\langle \mu, \beta_1^\vee \rangle > 0$. Therefore, there exists $\epsilon_1 \in \mathbb{R}_>$ such that

$$\langle \eta_0 + \epsilon_1 \eta_1, \beta_1^\vee \rangle = \langle \mu, \beta_1^\vee \rangle + \epsilon_1 \langle \eta_1, \beta_1^\vee \rangle > 0.$$

We find from (8) that

$$\eta_0 + \epsilon_1 \eta_1 = (1 + \epsilon_1)\mu - \epsilon_1 b_1 \beta_1 \in (1 + \epsilon_1)\mu - \mathbb{R}_>\{\beta_1\}.$$

Now let $2 \leq t \leq k$. By induction we have

$$x_{t-1} \in (1 + \epsilon_1 + \dots + \epsilon_{t-1})\mu - \mathbb{R}_>\{\beta_1, \dots, \beta_{t-1}\}. \quad (10)$$

Here $\langle \mu, \beta_t^\vee \rangle \geq 0$ and $\langle \beta_j, \beta_t^\vee \rangle \leq 0$ for all $j < t$. If $\beta_t \in J_>$ then $\langle \mu, \beta_t^\vee \rangle > 0$. If $\beta_t \notin J_>$ then there exists $s < t$ such that β_s, β_t are adjacent, from which we get $\langle \beta_s, \beta_t^\vee \rangle < 0$. We conclude that $\langle x_{t-1}, \beta_t^\vee \rangle > 0$. By induction we also have $\langle x_{t-1}, \beta_j^\vee \rangle > 0$ for all $j < t$. Hence we can choose $\epsilon_t \in \mathbb{R}_>$ such that

$$\langle x_t, \beta_j^\vee \rangle = \langle x_{t-1}, \beta_j^\vee \rangle + \epsilon_t \langle \eta_t, \beta_j^\vee \rangle > 0$$

for all $j = 1, \dots, t$. Furthermore, from (10) and (8) we find that $x_t = x_{t-1} + \epsilon_t \eta_t \in (1 + \epsilon_1 + \dots + \epsilon_t)\mu - \mathbb{R}_>\{\beta_1, \dots, \beta_t\}$.

Set $\epsilon := 1 + \epsilon_1 + \dots + \epsilon_k$. We have shown that

$$x_k = \eta_0 + \epsilon_1 \eta_1 + \dots + \epsilon_k \eta_k = \epsilon \mu - c_1 \beta_1 + \dots - c_k \beta_k$$

for some $c_1, \dots, c_k \in \mathbb{R}_>$, and that $\langle x_k, \alpha^\vee \rangle > 0$ for all $\alpha \in I$. For $\alpha \in \Pi \setminus I$ we find

$$\langle x_k, \alpha^\vee \rangle = \epsilon \langle \mu, \alpha^\vee \rangle - c_1 \langle \beta_1, \alpha^\vee \rangle - \dots - c_k \langle \beta_k, \alpha^\vee \rangle \geq 0$$

since $\langle \mu, \alpha^\vee \rangle \geq 0$ and $\langle \beta_j, \alpha^\vee \rangle \leq 0$ for $j = 1, \dots, k$. Furthermore, $\langle x_k, \alpha^\vee \rangle = 0$ if and only if $\langle \mu, \alpha^\vee \rangle = 0$ and $\langle \beta_j, \alpha^\vee \rangle = 0$ for $j = 1, \dots, k$. This is equivalent to $\alpha \in I_*$. Hence $\frac{1}{\epsilon} x_k \in \text{ri}(S) \cap (\mu - \mathbb{R}_>I) \cap C_{I_*}$. \square

In the following we set $\mathbb{R}_>\emptyset := \{0\}$.

Corollary 3.7 *Let $I \subseteq \Pi$ be μ -connected. Then the face F_I is fundamental with*

$$\text{ri}(F_I) \cap (\mu - \mathbb{R}_>I) \cap C_{I_*} \neq \emptyset. \quad (11)$$

The affine hull of F_I is $\mu + \mathbb{R}I$. In particular, $\dim F_I = |I|$.

Proof. The Corollary holds for $I = \emptyset$, where $F_\emptyset = \{\mu\}$ and $\emptyset_* = J_0$. Let $I \neq \emptyset$. Choose a simplex S as in Lemma 3.6. Then $S \subseteq F_I$ and $\mu + \mathbb{R}I = \text{aff}(S) \subseteq \text{aff}(F_I)$. Furthermore, $\text{aff}(F_I) \subseteq \mu + \mathbb{R}I$ by Theorem 3.3. Since S and F_I have the same affine hull $\mu + \mathbb{R}I$ we also obtain $\text{ri}(S) \subseteq \text{ri}(F_I)$. Thus (11) follows from (7). \square

Corollary 3.8 *The map $I \mapsto F_I$ from the set of all μ -connected subsets to the set of all nonempty fundamental faces $\mathcal{F} \setminus \{\emptyset\}$ is an isomorphism of partially ordered sets onto its image.*

Proof. Let I_1, I_2 be μ -connected. Clearly, $I_1 \subseteq I_2$ implies $F_{I_1} = \text{co}(W_{I_1}\mu) \subseteq \text{co}(W_{I_2}\mu) = F_{I_2}$. If $F_{I_1} \subseteq F_{I_2}$ then by Corollary 3.7 we obtain $\mu + \mathbb{R}I_1 \subseteq \mu + \mathbb{R}I_2$. Hence $I_1 \subseteq I_2$. \square

The following Proposition has been obtained when μ is an integral dominant weight and $I = \Pi$ in [10, Lemma 11.2] by Lie theoretic methods.

Proposition 3.9 *Let I be μ -connected. We have*

$$F_I \subseteq \bigsqcup_{\mu\text{-connected } K \subseteq I} \mu - \mathbb{R}_{>} K. \quad (12)$$

Proof. Clearly, the union is disjoint. If K_1, \dots, K_p are μ -connected and $r_1, \dots, r_p \in \mathbb{R}_{>}$ such that $r_1 + \dots + r_p = 1$ then $r_1(\mu - \mathbb{R}_{>} K_1) + \dots + r_p(\mu - \mathbb{R}_{>} K_p) \subseteq \mu - \mathbb{R}_{>}(K_1 \cup \dots \cup K_p)$, and $K_1 \cup \dots \cup K_p \subseteq I$ is μ -connected by Proposition 3.4. Therefore, it is sufficient to show that $W_I \mu$ is contained in the union on the right in (12).

Let $w \in W_I$. Then $w\mu \in \mu - \mathbb{R}_{>} K$ for some $K \subseteq I$. By Lemma 3.1 there exists $w_K \in W_K$ such that $w\mu = w_K \mu$. Inserting the decomposition $w_K = w^* w_*$ with $w^* \in W_{K^*}$ and $w_* \in W_{K_r} \subseteq W_{K^*}$ we get $w\mu = w_K \mu = w^* \mu \in \mu - \mathbb{R}_+ K^*$. It follows that $K \subseteq K^*$. Hence K is μ -connected. \square

When μ is an integral dominant weight the decomposition

$$H \cap \overline{C} = \bigsqcup_{\mu\text{-connected } K} (\mu - \mathbb{R}_{>} K) \cap \overline{C}, \quad (13)$$

which is a part of Remark 3.11, is a result of V. Kac and D. Peterson given in [10, Proposition 11.2 a)]. Its proof uses Lie theoretic and integral methods, which do not generalize to our situation.

For $\emptyset \neq K \subseteq \Pi$ we denote by K^0 the union of the connected components of K that are of finite type. We set $\emptyset^0 = \emptyset$.

Theorem 3.10 *If I is μ -connected then*

$$\text{ri}(F_I) \cap \overline{C} = (\mu - \mathbb{R}_{>} I) \cap \overline{C} = \bigsqcup_{I_f \subseteq I, (I_f)^0 = I_f} \underbrace{(\mu - \mathbb{R}_{>} I) \cap C_{I_* \cup I_f}}_{\neq \emptyset}.$$

Furthermore, we have

$$F_I \cap \overline{C} = \bigsqcup_{\mu\text{-connected } K \subseteq I} \text{ri}(F_K) \cap \overline{C}. \quad (14)$$

Remark 3.11 We obtain from the theorem for $I = \Pi^*$ the decomposition

$$H \cap \overline{C} = \bigsqcup_{\mu\text{-connected } K} \text{ri}(F_K) \cap \overline{C} = \bigsqcup_{\mu\text{-connected } K} (\mu - \mathbb{R}_{>} K) \cap \overline{C},$$

which will be key for the proofs of several of our results.

Proof. Because of $F_\emptyset = \{\mu\}$, $\text{ri}(F_\emptyset) = \{\mu\}$, $\mathbb{R}_{>}\emptyset = \{0\}$ by definition, and $\emptyset_* = J_0$, the theorem holds for $I = \emptyset$. We divide its proof for $I \neq \emptyset$ into several parts.

(i) We show that

$$(\mu - \mathbb{R}_{>} I) \cap \overline{C} = \bigsqcup_{I_f \subseteq I, (I_f)^0 = I_f} (\mu - \mathbb{R}_{>} I) \cap C_{I_* \cup I_f}.$$

The union on the right is disjoint and contained in the set on the left. To show the reverse inclusion let $\eta = \mu - \sum_{\alpha \in I} n_\alpha \alpha \in \overline{C}$ with $n_\alpha \in \mathbb{R}_{>}$ for all $\alpha \in I$. If $\beta \in \Pi \setminus I$ then

$$\langle \eta, \beta^\vee \rangle = \underbrace{\langle \mu, \beta^\vee \rangle}_{\geq 0} - \sum_{\alpha \in I} n_\alpha \underbrace{\langle \alpha, \beta^\vee \rangle}_{\leq 0} = 0$$

if and only if $\langle \mu, \beta^\vee \rangle = 0$ and $\langle \alpha, \beta^\vee \rangle = 0$ for all $\alpha \in I$. This is equivalent to $\beta \in I_*$.

Let K be a component of $\{\beta \in I \mid \langle \eta, \beta^\vee \rangle = 0\}$. For every $\beta \in K$ we have

$$\sum_{\alpha \in K} n_\alpha \langle \alpha, \beta^\vee \rangle = \underbrace{\langle \mu, \beta^\vee \rangle}_{\geq 0} - \sum_{\alpha \in I \setminus K} n_\alpha \underbrace{\langle \alpha, \beta^\vee \rangle}_{\leq 0} \geq 0. \quad (15)$$

From [10, Theorem 4.3] it follows that either K is of finite type, or that K is of affine type and in (15) we have equality for all $\beta \in K$. In the latter case we conclude that $\langle \mu, \beta^\vee \rangle = 0$ and $\langle \alpha, \beta^\vee \rangle = 0$ for all $\alpha \in I \setminus K$ and $\beta \in K$. Hence K is a component of I , which is contained in J_0 . However, this is impossible because every component of I is μ -connected.

(ii) We show that $(\mu - \mathbb{R}_{>} I) \cap C_{I_*} \subseteq F_I$ by contradiction. Let $\eta \in (\mu - \mathbb{R}_{>} I) \cap C_{I_*}$ and suppose that $\eta \notin F_I$. By Corollary 3.7 there exists

$$\eta_1 \in \text{ri}(F_I) \cap (\mu - \mathbb{R}_{>} I) \cap C_{I_*}.$$

Since $\mu \in F_I$ and $I_* \subseteq J_0$,

$$\eta_s := s\eta_1 + (1-s)\mu \in \text{ri}(F_I) \cap (\mu - \mathbb{R}_{>} I) \cap C_{I_*}$$

for all $0 < s \leq 1$. We choose some s such that $\eta_s - \eta \in \mathbb{R}_{>} I$, and consider the line segment

$$\overline{\eta_s \eta} \subseteq (\mu + \mathbb{R} I) \cap C_{I_*} \subseteq \{\eta' \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle \eta', \alpha^\vee \rangle > 0 \text{ for all } \alpha \in I\}.$$

Since $\eta_s \in \text{ri}(F_I) \subseteq F_I$, $\eta \notin F_I$, and F_I is convex, there exists some $\eta_b \in \overline{\eta_s \eta}$, $\eta_b \neq \eta_s$, such that $\overline{\eta_s \eta_b} \setminus \{\eta_b\} \subseteq F_I$ and $(\overline{\eta_b \eta} \setminus \{\eta_b\}) \cap F_I = \emptyset$.

Since $\overline{\eta_s \eta}$ is compact and I is finite there exists $\epsilon > 0$ such that $\langle \eta', \alpha^\vee \rangle \geq \epsilon$ for all $\eta' \in \overline{\eta_s \eta}$ and $\alpha \in I$. There exist $\eta_+ \in \overline{\eta_s \eta_b} \setminus \{\eta_b\}$ and $\eta_- \in \overline{\eta_b \eta} \setminus F_I$ such that

$$\eta_+ - \eta_- \in \left\{ \sum_{\alpha \in I} t_\alpha \alpha \mid 0 \leq t_\alpha \leq \frac{\epsilon}{|I|} \right\} \subseteq \left\{ \sum_{\alpha \in I} s_\alpha \langle \eta_+, \alpha^\vee \rangle \alpha \mid 0 \leq s_\alpha \leq \frac{1}{|I|} \right\}.$$

But then we get

$$\eta_- \in \eta_+ - \left\{ \sum_{\alpha \in I} s_\alpha \langle \eta_+, \alpha^\vee \rangle \alpha \mid s_\alpha \in \mathbb{R}^+, \sum_{\alpha \in I} s_\alpha \leq 1 \right\} = \text{co}(\eta_+, r_\alpha \eta_+ \mid \alpha \in I) \subseteq F_I,$$

which contradicts $\eta_- \notin F_I$.

(iii) We improve (ii) by showing that $(\mu - \mathbb{R}_{>} I) \cap C_{I_*} \subseteq \text{ri}(F_I)$. Let $\eta_0 \in (\mu - \mathbb{R}_{>} I) \cap C_{I_*}$. Let $\eta \in F_I$. For $\epsilon \in \mathbb{R}_+$ we set $\eta_\epsilon := \eta_0 + \epsilon(\eta - \eta_0)$. By Corollary 3.7 and the definition of I_* we get

$$\eta_0 - \eta \in \mathbb{R}I = \mathbb{R}I \cap \{ \eta' \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle \eta', \alpha^\vee \rangle = 0 \text{ for all } \alpha \in I_* \}.$$

Since $\mu - \mathbb{R}_{>} I$ is open in $\mu + \mathbb{R}I$, and C_{I_*} is open in $\{ \eta' \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle \eta', \alpha^\vee \rangle = 0 \text{ for all } \alpha \in I_* \}$ there exists $\epsilon_0 > 0$ such that $\eta_\epsilon \in (\mu - \mathbb{R}_{>} I) \cap C_{I_*}$ for all $0 \leq \epsilon \leq \epsilon_0$. From [5, Theorem 3.5] it follows that $\eta_0 \in \text{ri}(F_I)$.

(iv) Let $\emptyset \neq I_f \subseteq I$ such that $(I_f)^0 = I_f$. We show that $(\mu - \mathbb{R}_{>} I) \cap C_{I_* \cup I_f} \subseteq \text{ri}(F_I)$. Let $\eta_0 \in (\mu - \mathbb{R}_{>} I) \cap C_{I_* \cup I_f}$. From [10, Theorem 4.3 (Fin)] it follows that there exists $\gamma \in \mathbb{R}_{>} I_f$ such that $\langle \gamma, \alpha^\vee \rangle > 0$ for all $\alpha \in I_f$. For $t \in \mathbb{R}_+$ we set $\eta_t := \eta_0 + t\gamma$. By [32, Lemma 3.50] there exists $t_0 > 0$ such that

$$\eta_t \in C_{I_*} \quad \text{and} \quad \frac{1}{|W_{I_f}|} \sum_{w \in W_{I_f}} w\eta_t = \eta_0$$

for all $0 < t \leq t_0$. Since $\eta_0 \in \mu - \mathbb{R}_{>} I$ and $I_f \subseteq I$ we can choose $0 < t \leq t_0$ such that $\eta_t = \eta_0 + t\gamma \in \mu - \mathbb{R}_{>} I$. By (iii) we obtain $\eta_t \in \text{ri}(F_I)$. Because of $w\eta_t \in w\text{ri}(F_I) = \text{ri}(wF_I) = \text{ri}(F_I)$ for all $w \in W_{I_f}$, we find

$$\eta_0 = \frac{1}{|W_{I_f}|} \sum_{w \in W_{I_f}} w\eta_t \in \text{ri}(F_I).$$

(v) Let $I_f \subseteq I$ such that $(I_f)^0 = I_f$. We show that $(\mu - \mathbb{R}_{>} I) \cap C_{I_* \cup I_f} \neq \emptyset$. By Corollary 3.7 there exists $\eta \in (\mu - \mathbb{R}_{>} I) \cap C_{I_*}$. For $w \in W_{I_f}$ we get $w\eta \in \eta - \mathbb{R}_+ I_f \subseteq \mu - \mathbb{R}_{>} I$. From [32, Lemma 3.50] it follows that

$$\frac{1}{|W_{I_f}|} \sum_{w \in W_{I_f}} w\eta \in (\mu - \mathbb{R}_{>} I) \cap C_{I_* \cup I_f}.$$

(vi) From (i) and (iii), (iv) we find that

$$(\mu - \mathbb{R}_{>} I) \cap \overline{C} \subseteq \text{ri}(F_I) \cap \overline{C} \tag{16}$$

for all μ -connected sets I . From Proposition 3.9 and Corollary 3.8 it follows that

$$F_I \cap \overline{C} \subseteq \bigcup_{\mu\text{-connected } K \subseteq I} (\mu - \mathbb{R}_{>} K) \cap \overline{C} \subseteq \bigcup_{\mu\text{-connected } K \subseteq I} \text{ri}(F_K) \cap \overline{C} \subseteq F_I \cap \overline{C}.$$

Since the relative interiors of different faces are disjoint we have equality in (16). \square

The description of the following Corollary has been shown for $I = \Pi^*$ by E. Looijenga in [25, Corollary 1.14, Proposition 2.4] by a direct, involved proof. For integral dominant weights μ this description has been obtained in [10, Proposition 11.2 b)] from (13). Our proof of the Corollary follows that of V. Kac with more details.

Corollary 3.12 *Let I be μ -connected. If $(I \cap J_0)^0 = I \cap J_0$ then*

$$F_I \cap \overline{C} = (\mu - \mathbb{R}_+ I) \cap \overline{C} = (\mu - \mathbb{R}_+ I) \cap \overline{C}_{I_*}.$$

Proof. From Theorem 3.10 we get

$$F_I \cap \overline{C} = \bigsqcup_{\mu\text{-connected } K \subseteq I} (\mu - \mathbb{R}_> K) \cap \overline{C} \subseteq (\mu - \mathbb{R}_+ I) \cap \overline{C}.$$

To show the reverse inclusion let $\eta \in (\mu - \mathbb{R}_+ I) \cap \overline{C}$. Since $\mu \in F_I \cap \overline{C}$ we may assume $\eta \neq \mu$. There exists $\emptyset \neq K \subseteq I$ such that η is of the form $\eta = \mu - \sum_{\alpha \in K} n_\alpha \alpha \in \overline{C}$ with $n_\alpha \in \mathbb{R}_>$ for all $\alpha \in K$.

Suppose that K is not μ -connected. Then there exists a connected component L of K such that $L \subseteq K \cap J_0 \subseteq I \cap J_0$. For every $\beta \in L$ we find

$$0 \leq \langle \eta, \beta^\vee \rangle = \underbrace{\langle \mu, \beta^\vee \rangle}_{=0} - \sum_{\alpha \in L} n_\alpha \langle \alpha, \beta^\vee \rangle - \sum_{\alpha \in K \setminus L} n_\alpha \underbrace{\langle \alpha, \beta^\vee \rangle}_{=0}.$$

From [10, Theorem 4.3] it follows that L is of nonfinite type. This contradicts that L is contained in $I \cap J_0$, which is a union of components of finite type.

Obviously, $(\mu - \mathbb{R}_+ I) \cap \overline{C} \supseteq (\mu - \mathbb{R}_+ I) \cap \overline{C}_{I_*}$. The reverse inclusion follows by the definition of I_* . \square

That the statement (a) of the following Corollary implies the statement (c) is due to E. Looijenga in [25, Corollary 1.14, Corollary 2.5].

Corollary 3.13 *The following are equivalent.*

- (a) $(\Pi^* \cap J_0)^0 = \Pi^* \cap J_0$.
- (b) $H \cap \overline{C}$ is closed.
- (c) H is closed.

In particular, if the Kac-Moody algebra $\mathfrak{g}(A)$ is of finite, affine, or strongly hyperbolic type, then the orbit hull H is closed.

Proof. We first show the equivalence of (a) and (b). If we have $(\Pi^* \cap J_0)^0 = \Pi^* \cap J_0$ then $H \cap \overline{C} = (\mu - \mathbb{R}_+ \Pi^*) \cap \overline{C}$ by Corollary 3.12. Hence $H \cap \overline{C}$ is closed.

Now let $(\Pi^* \cap J_0)^0 \neq \Pi^* \cap J_0$, and choose a component K of $\Pi^* \cap J_0$ of nonfinite type. By [10, Theorem 4.3] there exists $\gamma \in \mathbb{R}_> K$ such that $\langle \gamma, \alpha^\vee \rangle \leq 0$ for all $\alpha \in K$. Clearly, $\langle \gamma, \alpha^\vee \rangle \leq 0$ for all $\alpha \in \Pi \setminus K$. By Theorem 3.10 there exists $\eta \in (\mu - \mathbb{R}_> \Pi^*) \cap \overline{C} = \text{ri}(H) \cap \overline{C}$. For $0 \leq t \leq 1$ we set $\tau_t := (1 - t)\mu + t\eta - \gamma$. Then

$$\tau_t \in (\mu - \mathbb{R}_> \Pi^*) \cap \overline{C} = \text{ri}(H) \cap \overline{C} \subseteq H \cap \overline{C}$$

for all $0 < t \leq 1$, and $\tau_0 = \mu - \gamma \in (\mu - \mathbb{R}_{>0}K) \cap \overline{C}$. Since K is not μ -connected it follows from Remark 3.11 that τ_0 is not contained in $H \cap \overline{C}$. Hence $H \cap \overline{C}$ is not closed.

Clearly, (c) implies (b). Now we suppose that (a) holds, and consider the free root base $\{\alpha^\vee \mid \alpha \in \Pi^*\} \subseteq \mathfrak{h}_\mathbb{R}$ and $\{\alpha \mid \alpha \in \Pi^*\} \subseteq \mathfrak{h}_\mathbb{R}^*$ for the Weyl group W_{Π^*} . Its fundamental chamber contains μ , and the isotropy group $W_{\Pi^* \cap J_0}$ of μ is finite. It follows from [25, Corollary 1.14, Corollary 2.5] that $H = F_{\Pi^*} = \text{co}(W_{\Pi^*} \mu)$ is closed, which is (c).

Let $\mathfrak{g}(A)$ be of finite, affine, or strongly hyperbolic type. If $J_0 = \Pi$ then $\Pi^* = \emptyset$. Thus $\Pi^* \cap J_0 = \emptyset$. If $J_0 \neq \Pi$ then $\Pi^* \cap J_0$ is a proper subset of Π . Hence it is either empty or a union of components of finite type. \square

Now we can improve Corollary 3.8.

Corollary 3.14 *The map $I \mapsto F_I$ from the set of all μ -connected subsets to the set $\mathcal{F} \setminus \{\emptyset\}$ of all nonempty fundamental faces is an isomorphism of partially ordered sets.*

Proof. By Corollary 3.8 it remains to show that the map is surjective. If F is a nonempty fundamental face then $\text{ri}(F) \cap \overline{C} \neq \emptyset$. By Remark 3.11 there exists some μ -connected set I such that $\text{ri}(F) \cap \text{ri}(F_I) \neq \emptyset$. Thus $F = F_I$. \square

To avoid case distinctions between the empty fundamental face and nonempty fundamental faces in the following theorem on the stabilizers and isotropy groups, and in many of the results of Section 4 we introduce some notation. For $F \in \mathcal{F}$ we set

$$\lambda^*(F) := \begin{cases} I & \text{if } F = F_I, I \text{ } \mu\text{-connected,} \\ \emptyset & \text{if } F = \emptyset. \end{cases} \quad (17)$$

$$\lambda_*(F) := \begin{cases} I_* & \text{if } F = F_I, I \text{ } \mu\text{-connected,} \\ \Pi & \text{if } F = \emptyset. \end{cases} \quad (18)$$

Furthermore, for $F \in \mathcal{F}$ we set

$$\lambda(F) := \lambda_*(F) \cup \lambda^*(F) = \begin{cases} I \cup I_* & \text{if } F = F_I, I \text{ } \mu\text{-connected,} \\ \Pi & \text{if } F = \emptyset. \end{cases} \quad (19)$$

The map $\lambda : \mathcal{F} \rightarrow 2^\Pi$ is called the *type map*.

Theorem 3.15 *If $F \in \mathcal{F}$ then*

$$W_*(F) = W_{\lambda_*(F)} \quad \text{and} \quad W(F) = W_{\lambda(F)} = W_{\lambda_*(F)} \times W_{\lambda^*(F)}.$$

In particular, the sets (18) and (19) can be obtained by

$$\{\alpha \in \Pi \mid r_\alpha \eta = \eta \text{ for all } \eta \in F\} = \lambda_*(F) \quad \text{and} \quad \{\alpha \in \Pi \mid r_\alpha F = F\} = \lambda(F).$$

Proof. Clearly, the theorem holds for $F = \emptyset$. Let $F = F_I$ where I is a μ -connected set. The elements of W_I and W_{I_*} commute because I and I_* are separated. Hence $W_{I \cup I_*} = W_I \times W_{I_*}$. Furthermore, since F_I is the convex hull of $W_I \mu$, and the elements of $W_{I_*} \subseteq W_{J_0}$ fix μ , we have $W_{I_*} \subseteq W_*(F_I) \subseteq W(F_I)$ and $W_I \subseteq W(F_I)$.

From Corollary 3.7 we find that there exists $\eta \in \text{ri}(F_I) \cap (\mu - \mathbb{R}_{>} I) \cap C_{I_*}$. The isotropy group of η is W_{I_*} by [10, Proposition 3.12 a)]. Hence $W_*(F_I) \subseteq W_{I_*}$.

Now let $w \in W(F_I)$. Then $w\eta \in F_I$. By Theorem 3.3 there exist $w' \in W_I$ and $\eta' \in F_I \cap \overline{C}$ such that $w\eta = w'\eta'$. Since every Weyl group orbit intersects \overline{C} in only one point we get $\eta' = \eta$. Thus $w\eta = w'\eta$, from which it follows that $w \in w'W_{I_*} \subseteq W_I W_{I_*} = W_{I \cup I_*}$. \square

Proposition 3.16 *Let I, I' be μ -connected and $w, w' \in W$. Then $wF_I \subseteq w'F_{I'}$ if and only if $I \subseteq I'$ and $w^{-1}w' \in W_{I_*}W_{I'} = W(F_I)W(F_{I'})$.*

Proof. If $I \subseteq I'$ then $I_* \supseteq I'_*$, and from Theorem 3.15 we obtain

$$W(F_I)W(F_{I'}) = W_{I \cup I_*}W_{I'_*}W_{I'} = W_{I \cup I_*}W_{I'} = W_{I_*}W_I W_{I'} = W_{I_*}W_{I'}.$$

By Corollary 3.7 we find that there exists $\eta \in \text{ri}(F_I) \cap (\mu - \mathbb{R}_{>} I) \cap C_{I_*}$. If $wF_I \subseteq w'F_{I'}$ then $(w')^{-1}w\eta \in F_{I'}$. By Theorem 3.3 there exist $\eta' \in F_{I'} \cap \overline{C}$ and $w'' \in W_{I'}$ such that $(w')^{-1}w\eta = w''\eta'$. Since every W -orbit intersects \overline{C} in only one point we get $\eta = \eta'$. Hence $(w')^{-1}w \in w''W_{\eta} \subseteq W_{I'}W_{I_*}$.

Now suppose that $(w')^{-1}w \in W_{I'}W_{I_*}$. Write $(w')^{-1}w = ab$ with $a \in W_{I'}$ and $b \in W_{I_*}$. Then $(w')^{-1}wF_I \subseteq F_{I'}$ is equivalent to $F_I = bF_I \subseteq a^{-1}F_{I'} = F_{I'}$ by Theorem 3.15, which in turn is equivalent to $I \subseteq I'$ by Corollary 3.14. \square

Corollary 3.17 *The set of fundamental faces \mathcal{F} is a cross-section for the action of W on $\mathcal{F}(H)$, that is, every face of H is W -equivalent to a unique fundamental face of H .*

Proof. The face $\emptyset \in \mathcal{F}(H)$ is W -equivalent only to $\emptyset \in \mathcal{F}$. Let $F \in \mathcal{F}(H) \setminus \{\emptyset\}$ and $\eta_0 \in \text{ri}(F)$. There exists $w \in W$ such that $w\eta_0 \in \overline{C}$. By Remark 3.11 there exists some μ -connected set K such that $\text{ri}(F_K) \ni w\eta_0 \in w \text{ri}(F) = \text{ri}(wF)$. Hence $wF = F_K$.

If also $w'F = F_{K'}$ for $w' \in W$ and some μ -connected set K' , then $w^{-1}F_K = (w')^{-1}F_{K'}$. From Proposition 3.16 we get $K = K'$. \square

We need also the following refinement which follows immediately from Proposition 3.16 and Corollary 3.17.

Corollary 3.18 *Let F be a fundamental face. Then every face contained in F is $W(F)$ -equivalent to a unique fundamental face contained in F .*

From the Corollaries 3.17, 3.14, and 3.7 we obtain immediately, that the edges of H which contain the vertex μ are given by

$$\overline{\mu r_\alpha \mu} \quad \text{where} \quad \alpha \in W_{J_0} J_{>} = \bigsqcup_{\beta \in J_{>}} W_{J_0} \beta.$$

Corollary 3.19 *The following are equivalent.*

- (a) $(\Pi^* \cap J_0)^0 = \Pi^* \cap J_0$.
- (b) *There are only finitely many edges of H that contain the vertex μ .*

In particular, if the Kac-Moody algebra $\mathfrak{g}(A)$ is of finite, affine, or strongly hyperbolic type, then there are only finitely many edges of H which contain the vertex μ

Proof. We have $\Pi = \Pi^* \cup \Pi_*$ and Π^*, Π_* are separated with $J_{>} \subseteq \Pi^*, \Pi_* \subseteq J_0$. Hence

$$W_{J_0} J_{>} = W_{(\Pi^* \cap J_0) \cup \Pi_*} J_{>} = W_{\Pi^* \cap J_0} W_{\Pi_*} J_{>} = W_{\Pi^* \cap J_0} J_{>}.$$

If $(\Pi^* \cap J_0)^0 = \Pi^* \cap J_0$ then $W_{\Pi^* \cap J_0}$ is finite. Thus $W_{\Pi^* \cap J_0} J_{>}$ is finite. Now let $(\Pi^* \cap J_0)^0 \neq \Pi^* \cap J_0$. We show that $W_{\Pi^* \cap J_0} J_{>}$ is infinite.

There exists a component K of $\Pi^* \cap J_0$ of nonfinite type. Let K' be the component of Π^* that contains K . Since K' is connected and μ -connected there exists $\beta \in K' \cap J_{>}$ adjacent to some $\gamma \in K$. Now $\{\alpha^\vee \mid \alpha \in K\} \subseteq \mathfrak{h}_{\mathbb{R}}$ and $\{\alpha \mid \alpha \in K\} \subseteq \mathfrak{h}_{\mathbb{R}}^*$ is a free root base for the Weyl group W_K . Moreover, $-\beta$ is in its fundamental chamber in $\mathfrak{h}_{\mathbb{R}}^*$, since $-\langle \beta, \alpha^\vee \rangle \geq 0$ for all $\alpha \in K$. Hence the isotropy group of β in W_K is given by W_L where $L = \{\alpha \in K \mid r_\alpha \beta = \beta\}$. Furthermore, $L \subsetneq K$ since $\gamma \notin L$. By [1, Proposition 2.43] the set $W_K \beta$ is infinite.

Let $\mathfrak{g}(A)$ be of finite, affine, or strongly hyperbolic type. If $J_0 = \Pi$ then $\Pi^* = \emptyset$. Thus $\Pi^* \cap J_0 = \emptyset$. If $J_0 \neq \Pi$ then $\Pi^* \cap J_0$ is a proper subset of Π . Hence it is either empty or a union of components of finite type. \square

We define the map $\text{red} : W \rightarrow \Pi$ as follows: We set $\text{red}(1) := \emptyset$. If $w \in W$ and $w = r_{i_1} r_{i_1} \cdots r_{i_k}$ is a reduced expression we set $\text{red}(w) := \{\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_k}\}$, which is independent of the chosen reduced expression by [1, Theorem 2.33].

Theorem 3.20 (a) Let I, I' be μ -connected and $w \in {}^I W^{I'}$. Then

$$F_I \cap w F_{I'} = \begin{cases} F_{(I \cap w I')^*} & \text{if } w \in W_{J_0}, \\ \emptyset & \text{if } w \notin W_{J_0}. \end{cases}$$

(b) Let I, I' be μ -connected and $w \in {}^{I^*} W^{I'_*}$. Then $I \cup I' \cup \text{red}(w)$ is μ -connected and

$$F_I \vee w F_{I'} = F_{I \cup I' \cup \text{red}(w)}.$$

Remark 3.21 The lattice intersection and lattice join of two arbitrary nonempty faces can be reduced to Theorem 3.20. Use ${}^{I \cup I^*} W^{I' \cup I'_*} \subseteq {}^I W^{I'}, {}^{I^*} W^{I'_*}$ and Theorem 3.15.

Proof. We first show (a). From Lemma 2.6 and Theorem 3.3 we find that the face $F_I \cap w F_{I'}$ is generated by

$$(F_I \cap w F_{I'}) \cap W\mu = (F_I \cap W\mu) \cap (w F_{I'} \cap W\mu) = W_I \mu \cap w W_{I'} \mu.$$

Suppose that $F_I \cap w F_{I'} \neq \emptyset$. Let $w_1 \in W_I, w'_1 \in W_{I'}$ such that $w_1 \mu = w w'_1 \mu$. Then $(w_1)^{-1} w w'_1 \in W_{J_0}$. From [1, Proposition 2.23] we find that there exist $w_2 \in W_I, w'_2 \in W_{I'}$ such that

$$(w_2)^{-1} w w'_2 = (w_1)^{-1} w w'_1 \in W_{J_0} \quad \text{and} \quad l((w_2)^{-1} w w'_2) = l((w_2)^{-1}) + l(w) + l(w'_2).$$

It follows that $w \in W_{J_0}$, $w_2 \in W_{J_0} \cap W_I = W_{J_0 \cap I}$, and $w'_2 \in W_{J_0} \cap W_{I'} = W_{J_0 \cap I'}$. Therefore, we obtain

$$W_I \ni w_2(w_1)^{-1} = ww'_2(w'_1)^{-1}w^{-1} \in wW_{I'}w^{-1}.$$

Hence $w_2(w_1)^{-1} \in W_{I \cap wI'}$ by [1, Lemma 2.25], and $w_1\mu \in W_{I \cap wI'}w_2\mu = W_{I \cap wI'}\mu$. Thus $F_I \cap wF_{I'} \subseteq F_{I \cap wI'} = F_{(I \cap wI')^*}$.

We have $W_{I \cap wI'} = W_I \cap wW_{I'}w^{-1}$ by [1, Lemma 2.25]. If $w \in W_{J_0}$ then

$$W_{I \cap wI'}\mu \subseteq W_I\mu \cap wW_{I'}w^{-1}\mu = W_I\mu \cap wW_{I'}\mu.$$

Hence, $F_{(I \cap wI')^*} = F_{I \cap wI'} \subseteq F_I \cap wF_{I'}$.

We now prove (b). Set $J := I \cup I' \cup \text{red}(w)$. Since $w \in W_J = W_{J^*}W_{J_r}$ with $J_r \subseteq J_*$ we can write $w = w^*w_*$ with $w^* \in W_{J^*}$ and $w_* \in W_{J_*}$. Because of $J_* \subseteq I'_*$ we get

$$W_{I_*}wW_{I'_*} = W_{I_*}w^*w_*W_{I'_*} = W_{I_*}w^*W_{I'_*}.$$

Since w is a minimal double coset representative we find $l(w^*) \geq l(w) = l(w^*) + l(w_*)$. We conclude that $l(w_*) = 0$ and $w_* = 1$. Thus $\text{red}(w) = \text{red}(w^*) \subseteq J^*$. Hence $J = I \cup I' \cup \text{red}(w) \subseteq J^*$, which shows that J is μ -connected.

By Corollary 3.17 and Corollary 3.14 there exist $w_1 \in W$ and some μ -connected set K such that $F_I \vee wF_{I'} = w_1F_K$.

Clearly, $F_I, wF_{I'} \subseteq F_{I \cup I' \cup \text{red}(w)}$. Hence $w_1F_K = F_I \vee wF_{I'} \subseteq F_{I \cup I' \cup \text{red}(w)}$. From Proposition 3.16 we get $K \subseteq I \cup I' \cup \text{red}(w)$.

Since $F_I, wF_{I'} \subseteq w_1F_K$ we find from Proposition 3.16 that $I, I' \subseteq K$, and $w_1 \in W_{I_*}W_K$, $w^{-1}w_1 \in W_{I'_*}W_K$. Eliminating w_1 , we obtain $w \in W_{I_*}W_KW_{I'_*}$, and equivalently $W_{I_*}wW_{I'_*} \cap W_K \neq \emptyset$. It follows from [1, Proposition 2.23], similarly as in the proof of part (a), that $w \in W_K$. We conclude that $I \cup I' \cup \text{red}(w) \subseteq K$.

We have shown that $w_1F_K \subseteq F_{I \cup I' \cup \text{red}(w)}$, and $K = I \cup I' \cup \text{red}(w)$, from which it follows that the dimension of both faces coincide. Therefore, $w_1F_K = F_{I \cup I' \cup \text{red}(w)}$. \square

Let the partially ordered set (L, \leq) be a lattice. In this article a subset $S \subseteq L$ is called a sublattice of L , if the partially ordered set (S, \leq) is a lattice with the same smallest and biggest elements, and the same lattice operations as in L .

Corollary 3.22 *The set \mathcal{F} of fundamental faces is a sublattice of the face lattice $\mathcal{F}(H)$.*

Proof. Trivially, $\emptyset, H \in \mathcal{F}$. Let $F_1, F_2 \in \mathcal{F}$. We obtain from Theorem 3.20 that

$$F_1 \cap F_2 \in \mathcal{F} \quad \text{and} \quad F_1 \vee F_2 \in \mathcal{F} \tag{20}$$

for all $F_1, F_2 \in \mathcal{F} \setminus \{\emptyset\}$. Clearly, (20) holds if $F_1 = \emptyset$ or $F_2 = \emptyset$. \square

4 The monoid $M(\rho)$

In this section we fix an irreducible highest weight representation (V, ρ) of the Kac-Moody algebra $\mathfrak{g}(A)$ of highest weight $\mu \in P^+$. We assume that Π is μ -connected, i.e., no connected component of Π is contained in the type J_0 of μ . We denote by (V, ρ) the corresponding irreducible highest weight representation of the Kac-Moody group \mathbf{G} . We fix a contravariant nondegenerate symmetric bilinear form $(|)$ on V .

4.1 The definition of $M(\rho)$

The linear space V decomposes into a direct sum of weight spaces

$$V = \bigoplus_{\eta \in P(V)} V_\eta.$$

The weight hull H of ρ is the convex hull of the set of weights $P(V) \subset \mathfrak{h}_\mathbb{R}^*$. By [10, Proposition 11.3 a)] H coincides with the orbit hull of μ .

If F is a face of H we get a decomposition

$$V = V_F \oplus V_F^\perp \quad \text{with} \quad V_F := \bigoplus_{\eta \in F \cap P(V)} V_\eta \quad \text{and} \quad V_F^\perp := \bigoplus_{\eta \in P(V) \setminus F} V_\eta \quad (21)$$

which is also orthogonal with respect to the nondegenerate bilinear form $(|)$. We call V_F the *face vector space* associated to the face F . We denote by $e(F)$ the corresponding linear projector defined by

$$e(F)v = \begin{cases} v, & \text{if } v \in V_F, \\ 0, & \text{if } v \in V_F^\perp. \end{cases} \quad (22)$$

The projector $e(F)$ is an idempotent of $\text{End}(V)$. Its adjoint $e(F)^\star$ with respect to the nondegenerate bilinear form $(|)$ exists and we have $e(F)^\star = e(F)$.

The set

$$E := \{e(F) \mid F \in \mathcal{F}(H)\}$$

is a commutative submonoid of $\text{End}(V)$ consisting of idempotents. Its multiplication is given by

$$e(F)e(F') = e(F \cap F') \quad \text{where } F, F' \in \mathcal{F}(H). \quad (23)$$

Moreover, $e(\emptyset)$ is the zero and $e(H)$ is the identity of E as well as of $\text{End}(V)$.

Recall that $G := \mathbb{C}^\times \rho(G)$. We define the monoid $M(\rho)$ to be the submonoid of $\text{End}(V)$ generated by G and E , that is,

$$M(\rho) := \langle G, E \rangle.$$

We often write M instead of $M(\rho)$. The adjoints of the elements of M with respect to the nondegenerate bilinear form $(|)$ exist and are contained in M . In this way we get an anti-involution \star on M extending the Chevalley anti-involution on G .

We define \overline{T} to be the submonoid of M generated by T and E . We define \overline{N} to be the submonoid of M generated by N and E . To describe these submonoids we need the following Lemma.

Lemma 4.1 *Let F be a face of H , and let $n \in N$ represent $w \in W$. Then*

$$ne(F)n^{-1} = e(wF).$$

Proof. This is straightforward from (22) and (2). □

If Y is a submonoid of M we denote by Y^\times its unit group, and by $E(Y)$ its set of idempotents.

Proposition 4.2 (a) \overline{T} is a unit regular commutative monoid with $\overline{T}^\times = T$ and $E(\overline{T}) = E$. In particular, $\overline{T} = TE = ET$.

(b) \overline{N} is a unit regular inverse monoid with $\overline{N}^\times = N$ and $E(\overline{N}) = E$. In particular, $\overline{N} = NE = EN$.

Proof. From Lemma 4.1 it follows that \overline{T} is commutative. Hence $\overline{T} = TE = ET$. From this Lemma we also find that $\overline{N} = NE = EN$.

Clearly, $T \subseteq \overline{T}^\times$. Conversely, $x \in \overline{T}^\times$ is of the form $x = te$ for some $t \in T$ and $e \in E$. Therefore, $e = t^{-1}x \in E(\overline{T}) \cap T^\times = \{1\}$. Thus $x = t \in T$. Similarly, we get $\overline{N}^\times = N$.

We have $E \subseteq E(\overline{T}) \subseteq E(\overline{N})$. Conversely, any idempotent $x \in E(\overline{N})$ can be written in the form $x = n_w e(F)$ for some $n_w \in N$ projecting to $w \in W$, and some face F of H . From $x = x^2$ we get $e(F) = e(F)n_w e(F)$. Evaluating at $v_\eta \in V_\eta \setminus \{0\}$ where $\eta \in F$ we find $0 \neq v_\eta = e(F)n_w v_\eta$. Since $n_w v_\eta$ is homogeneous it follows that $v_\eta = n_w v_\eta$. Hence $xv_\eta = n_w v_\eta = v_\eta$. For $v_\eta \in V_\eta$ where $\eta \notin F$ we obtain $xv_\eta = 0$. We conclude that $x = e(F) \in E$.

Clearly, \overline{T} and \overline{N} are unit regular monoids. The idempotents of \overline{N} commute. From [9, Theorem 5.1.1] it follows that \overline{N} is an inverse monoid. \square

Our aim below is to establish the Bruhat decomposition, Tits system, and the unit regularity for M . To this end, we need intensive preparations. We proceed similarly as in [27, Chapter 2].

4.2 Weight strings

Let F be a face of H . The weights in $P(V) \cap F$ are called F -weights. We investigate explicitly the α -weight string through an F -weight where α is a real root.

Recall from Section 3 the definition and the properties of the parabolic subgroups $W(F)$, $W_*(F)$. We define

$$\begin{aligned}\Delta(F) &:= \{\alpha \in \Delta^{re} \mid r_\alpha \in W(F)\} = \{\alpha \in \Delta^{re} \mid r_\alpha F = F\}, \\ \Delta_*(F) &:= \{\alpha \in \Delta^{re} \mid r_\alpha \in W_*(F)\} = \{\alpha \in \Delta^{re} \mid r_\alpha \eta = \eta \text{ for all } \eta \in F\}, \\ \Delta^*(F) &:= \Delta(F) \setminus \Delta_*(F).\end{aligned}$$

Note that if $F = wF'$ with $w \in W$ and F' a fundamental face then

$$\begin{aligned}\Delta(F) &= w\Delta(F') = wW_{\lambda(F')}\lambda(F'), \\ \Delta_*(F) &= w\Delta_*(F') = wW_{\lambda_*(F')}\lambda_*(F'), \\ \Delta^*(F) &= w\Delta^*(F') = wW_{\lambda^*(F')}\lambda^*(F').\end{aligned}$$

Since the isotropy group $W(F')$ of F' leaves $\Delta_\pm^{re} \setminus \Delta(F')$ invariant we can define

$$\Delta_p(F) := w(\Delta_+^{re} \setminus \Delta(F')) \quad \text{and} \quad \Delta_n(F) := w(\Delta_-^{re} \setminus \Delta(F')).$$

In particular, we have $\Delta(\emptyset) = \Delta_*(\emptyset) = \Delta^{re}$, $\Delta^*(\emptyset) = \Delta_p(\emptyset) = \Delta_n(\emptyset) = \emptyset$ and $\Delta(H) = \Delta^*(H) = \Delta^{re}$, $\Delta_*(H) = \Delta_p(H) = \Delta_n(H) = \emptyset$.

Lemma 4.3 *Let F be a face of H , and $w \in W$. Then:*

- (a) $w \in W(F)$ if and only if $w(P(V) \cap F) = P(V) \cap F$.
- (b) $w \in W_*(F)$ if and only if $w\eta = \eta$ for all $\eta \in P(V) \cap F$.

Proof. The Lemma holds trivially for $F = \emptyset$. Since $P(V)$ is W -invariant it is sufficient to show the Lemma for a nonempty fundamental face F . From

$$W_{\lambda^*(F)}\mu \subseteq P(V) \cap F \subseteq F = \text{co}(W_{\lambda^*(F)}\mu)$$

we get $F = \text{co}(P(V) \cap F)$, from which the Lemma follows immediately. \square

Theorem 4.4 *Let F be a nonempty face of H .*

- (a) *If $\alpha \in \Delta(F)$ then for every F -weight η the α -weight string through η lies completely in F .*
- (b) *If $\alpha \in \Delta_*(F)$ then for every F -weight η the α -weight string through η has only one element. In particular, $\langle \eta, \alpha^\vee \rangle = 0$.*
- (c) *If $\alpha \in \Delta^*(F)$ then there exists an F -weight η such that the α -weight string through η has more than one element. In particular, there exist F -weights η_+, η_- such that $\langle \eta_+, \alpha^\vee \rangle > 0$ and $\langle \eta_-, \alpha^\vee \rangle < 0$.*

Proof. To prove (a) let η be an F -weight. If the α -weight string through η has only one element it is contained in F . Suppose that the string has more than one element. If η is one of the two ends of the weight string, then $r_\alpha\eta$ is the other end. Moreover, $r_\alpha\eta$ is an F -weight because of $\alpha \in \Delta(F)$. Since F is convex it follows that the full string is in F . If η is not an end of the α -weight string through η then the string is of the form

$$\eta - p\alpha, \dots, \eta, \dots, \eta + q\alpha, \quad \text{with } p, q \geq 1.$$

Since F is a face of the weight hull H the whole string is contained in F .

To prove (b) let η be an F -weight. The α -weight string through η is contained in F by (a), and the reflection r_α interchanges its ends. Since $\alpha \in \Delta_*(F)$, the reflection r_α fixes its ends. Hence the α -weight string through η has only one element.

We now prove (c). Since $r_\alpha \notin W_*(F)$ it follows from Lemma 4.3 (b) that $r_\alpha\eta \neq \eta$ for some F -weight η . Hence the α -weight string through η has more than one element, and is contained in F by (a). Choose η_+ be the end of the string with $\langle \eta_+, \alpha^\vee \rangle > 0$, and η_- to be the end of the string with $\langle \eta_-, \alpha^\vee \rangle < 0$. \square

Corollary 4.5 *Let F be a nonempty face of H and $\alpha \in \Delta(F)$. If $\eta \in P(V)$ is not an F -weight then no weight in the α -weight string through η is an F -weight.*

Theorem 4.6 *Let F be a nonempty face of H .*

- (a) *If $\alpha \in \Delta_p(F)$ and η is an F -weight then $\langle \eta, \alpha^\vee \rangle \geq 0$. The α -weight string through η is*

$$\eta, \eta - \alpha, \dots, r_\alpha(\eta) = \eta - \langle \eta, \alpha^\vee \rangle \alpha,$$

and η is the only F -weight in the string. In particular, $\eta + \alpha$ is not a weight. There exists an F -weight η_+ such that $\langle \eta_+, \alpha^\vee \rangle > 0$.

(b) If $\alpha \in \Delta_n(F)$ and η is an F -weight then $\langle \eta, \alpha^\vee \rangle \leq 0$. The α -weight string through η is

$$\eta, \eta + \alpha, \dots, r_\alpha(\eta) = \eta - \langle \eta, \alpha^\vee \rangle \alpha,$$

and η is the only F -weight in the string. In particular, $\eta - \alpha$ is not a weight. There exists an F -weight η_- such that $\langle \eta_-, \alpha^\vee \rangle < 0$.

Proof. We only prove (a), the proof of (b) is similar. We have $F = wF'$ for some $w \in W$ and a nonempty fundamental face F' , and η, α can be written as $\eta = w\eta'$ with $\eta' \in F'$, and $\alpha = w\alpha'$ with $\alpha' \in \Delta_+^{re} \setminus \Delta(F')$. Then $r_\alpha\eta = w(r_{\alpha'}\eta')$, and the α -weight string through η and the α' -weight string through η' are related by

$$P(V) \cap (\eta + \mathbb{Z}\alpha) = w(P(V) \cap (\eta' + \mathbb{Z}\alpha')).$$

Moreover, $\langle \eta, \alpha^\vee \rangle = \langle \eta', w^{-1}\alpha^\vee \rangle = \langle \eta', (w^{-1}\alpha)^\vee \rangle = \langle \eta', \alpha'^\vee \rangle$. Therefore, it is sufficient to prove (a) for a fundamental face F .

We first show that $\langle \eta, \alpha^\vee \rangle \geq 0$ for all $\eta \in F$. Since $F = \text{co}(W_{\lambda(F)}\mu)$, it suffices to show this for all $\eta \in W_{\lambda(F)}\mu$. Let $w_1 \in W_{\lambda(F)}$. Then $w_1^{-1}\alpha \in w_1^{-1}(\Delta_+^{re} \setminus \Delta(F)) \subseteq \Delta_+^{re}$ and $(w_1^{-1}\alpha)^\vee > 0$. Hence

$$\langle w_1\mu, \alpha^\vee \rangle = \langle \mu, w_1^{-1}\alpha^\vee \rangle = \langle \mu, (w_1^{-1}\alpha)^\vee \rangle \geq 0.$$

We now describe the α -weight string through the F -weight η when $\langle \eta, \alpha^\vee \rangle > 0$. Suppose that η is not an end of the string. Since F is a face of the weight hull H , the whole string is contained in F . In particular $r_\alpha\eta$ is an F -weight. But this is not possible because of $\langle r_\alpha\eta, \alpha^\vee \rangle = -\langle \eta, \alpha^\vee \rangle < 0$. Hence the weight string is of the form

$$\eta, \eta - \alpha, \dots, r_\alpha(\eta) = \eta - \langle \eta, \alpha^\vee \rangle \alpha.$$

In particular, $\eta + \alpha$ is not a weight.

We next describe the α -weight string through the F -weight η when $\langle \eta, \alpha^\vee \rangle = 0$. Clearly, $r_\alpha(\eta) = \eta \in F \cap r_\alpha F$. The face $F_1 := F \cap r_\alpha F$ is fixed pointwise by r_α . Hence $\alpha \in \Delta_*(F_1)$. By Theorem 4.4 (b) the α -weight string through η consists of η only.

Suppose that there does not exist an F -weight η_+ such that $\langle \eta_+, \alpha^\vee \rangle > 0$. Then $\langle \eta, \alpha^\vee \rangle = 0$ for all F -weights η . Lemma 4.3 (b) implies that $\alpha \in \Delta_*(F)$, which contradicts $\alpha \in \Delta_p(F) = \Delta_+^{re} \setminus \Delta(F)$. \square

4.3 Isotropy monoids, isotropy groups, and stabilizers

Let U be a linear subspace of V , and let Y be a subgroup of G . The *left isotropy monoid* of U in Y , which is a submonoid of Y , is defined by

$$N_Y^\subseteq(U) := \{y \in Y \mid yU \subseteq U\}.$$

The *isotropy group* of U in Y , which is a subgroup of Y , is defined by

$$N_Y(U) := \{y \in Y \mid yU = U\}.$$

The *stabilizer* of U in Y , which is a normal subgroup of $N_Y(U)$, is defined by

$$Z_Y(U) := \{y \in Y \mid yv = v \text{ for all } v \in U\}.$$

We have $N_Y(U) = N_Y^\subseteq(U) \cap N_Y^\subseteq(U)^{-1}$. If $N_Y^\subseteq(U)$ is a group then $N_Y(U) = N_Y^\subseteq(U)$. We denote by U^\perp the biggest subspace of V orthogonal to U with respect to (\mid) . It is easy to see that if $V = U \oplus U^\perp$ then

$$N_Y^\subseteq(U^\perp) = N_{Y^\star}^\subseteq(U)^\star \quad \text{and} \quad N_Y(U^\perp) = N_{Y^\star}(U)^\star. \quad (24)$$

For $g \in G$ we have $N_Y^\subseteq(gU) = gN_{g^{-1}Yg}^\subseteq(U)g^{-1}$, and $N_Y(gU) = gN_{g^{-1}Yg}(U)g^{-1}$, and $Z_Y(gU) = gZ_{g^{-1}Yg}(U)g^{-1}$. Trivially, we have

$$Z_Y(\{0\}) = N_Y(\{0\}) = N_Y^\subseteq(\{0\}) = Y \quad \text{and} \quad Z_Y(V) = \{1\}, \quad N_Y(V) = N_Y^\subseteq(V) = Y. \quad (25)$$

One of the reasons to introduce these concepts is the following proposition, which describes how to perform certain calculations with the idempotents in E . These are basic for the investigation of M .

Proposition 4.7 *Let F, F' be faces of H . Let Y be a subgroup of G , and $y \in Y$. Then*

$$\begin{aligned} \text{(a)} \quad ye(F) &= e(F)ye(F) \Leftrightarrow y \in N_Y^\subseteq(V_F). \\ \text{(b)} \quad e(F)y &= e(F)ye(F) \Leftrightarrow y \in N_{Y^\star}^\subseteq(V_F)^\star. \end{aligned}$$

In addition,

$$\text{(c)} \quad e(F) = ye(F)y^{-1} \Leftrightarrow y \in N_Y^\subseteq(V_F) \cap N_{Y^\star}^\subseteq(V_F)^\star = N_Y(V_F) \cap N_{Y^\star}(V_F)^\star.$$

Furthermore, we have

$$\begin{aligned} \text{(d)} \quad ye(F) &= e(F') \Leftrightarrow F' = F \quad \text{and} \quad y \in Z_Y(V_F). \\ \text{(e)} \quad e(F)y &= e(F') \Leftrightarrow F' = F \quad \text{and} \quad y \in Z_{Y^\star}(V_F)^\star. \end{aligned}$$

Proof. It is straightforward to see that (b) can be obtained from (a) by applying the Chevalley anti-involution: We have $e(F)y = e(F)ye(F)$ if and only if $y^\star e(F) = e(F)y^\star e(F)$, if and only if $y^\star \in N_{Y^\star}(V_F)$ by (a) for the group Y^\star . Similarly, (e) can be obtained from (d).

We now prove (a). We have $ye(F) = e(F)ye(F)$ if and only if $ye(F)v = e(F)ye(F)v$ for all $v \in V$, if and only if $yv = e(F)yv$ for all $v \in V_F$, if and only if $yv \in V_F$ for all $v \in V_F$, in other words, $y \in N_Y^\subseteq(V_F)$.

We next prove (c). By (a), (b) we obtain $y \in N_Y^\subseteq(V_F) \cap N_{Y^\star}^\subseteq(V_F)^\star$ if and only if $ye(F) = e(F)ye(F)$ and $e(F)y = e(F)ye(F)$, if and only if $ye(F) = e(F)y$.

Two linear projectors coincide if their images and their kernels coincide. Therefore, $ye(F)y^{-1} = e(F)$ if and only if $yV_F = V_F$ and $yV_F^\perp = V_F^\perp$, if and only if $y \in N_Y(V_F)$ and $y \in N_Y(V_F^\perp) = N_{Y^\star}(V_F)^\star$ by (24).

To prove (d) let $ye(F) = e(F')$. Comparing the kernels we get $V_F^\perp = V_{F'}^\perp$, from which it follows that $F = F'$. We have $ye(F) = e(F)$ if and only if $ye(F)v = e(F)v$ for all $v \in V$, if and only if $yv = v$ for all $v \in V_F$, that is, $y \in Z_Y(V_F)$. \square

Our next aim is to determine $N_G^\subseteq(V_F)$, $N_G(V_F)$, and $Z_G(V_F)$, which is not straightforward. As intermediate steps we determine $N_Y^\subseteq(V_F)$, $N_Y(V_F)$, and $Z_Y(V_F)$ for $Y = U_\alpha$, $\alpha \in \Delta^{re}$, for $Y = T$, and for $Y = N$.

For $v = \sum_\eta v_\eta \in V$ with $v_\eta \in V_\eta$, $\eta \in P(V)$, we define

$$\text{supp}(v) := \{\eta \in P(V) \mid v_\eta \neq 0\}.$$

In addition to the results on the weight strings of Theorems 4.4 and 4.6 we need the following easy Lemma to determine $N_{U_\alpha}^\subseteq(V_F)$, $N_{U_\alpha}(V_F)$, and $Z_{U_\alpha}(V_F)$, $\alpha \in \Delta^{re}$.

Lemma 4.8 *Let $\eta \in P(V)$ and $\alpha \in \Delta^{re}$.*

- (a) *For all $v_\eta \in V_\eta$ and $u \in U_\alpha$ we have $\text{supp}(uv_\eta) \subseteq P(V) \cap (\eta + \mathbb{Z}_+\alpha)$.*
- (b) *There exists a weight vector $v_\eta \in V_\eta$ such that for all $u \in U_\alpha \setminus \{1\}$ we have $\text{supp}(uv_\eta) = P(V) \cap (\eta + \mathbb{Z}_+\alpha)$.*

Proof. We first prove (a). If $v_\eta \in V_\eta$ and $u = \rho(\exp x_\alpha)$ for some $x_\alpha \in \mathfrak{g}_\alpha$ then

$$uv_\eta = \rho(\exp x_\alpha)v_\eta = \sum_{j \in \mathbb{Z}_+, \eta+j\alpha \in P(V)} v_{\eta+j\alpha} \quad \text{with} \quad v_{\eta+j\alpha} = \frac{1}{j!} \rho(x_\alpha)^j v_\eta \in V_{\eta+j\alpha}.$$

Hence, $\text{supp}(uv_\eta) \subseteq P(V) \cap (\eta + \mathbb{Z}_+\alpha)$.

We now prove (b). Choose $x_\alpha \in \mathfrak{g}_\alpha$, $x_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $[x_\alpha, x_{-\alpha}] = \alpha^\vee$. Then $\mathfrak{s}_\alpha := \mathbb{C}x_\alpha + \mathbb{C}\alpha^\vee + \mathbb{C}x_{-\alpha}$ is a Lie subalgebra of \mathfrak{g} isomorphic to $sl(2, \mathbb{C})$. Since V is integrable, it decomposes into a direct sum of irreducible finite-dimensional \mathfrak{s}_α -modules, whose $\mathbb{C}\alpha^\vee$ -weight spaces are also \mathfrak{h} -weight spaces. The α -weight string through η is finite. Hence, among these modules exists a module D , whose set $P(D)$ of \mathfrak{h} -weights coincides with $P(V) \cap (\eta + \mathbb{Z}\alpha)$, the α -weight string through η . Choose $v_\eta \in D_\eta \setminus \{0\}$. Then $\text{supp}(\rho(\exp cx_\alpha)v_\eta) = P(V) \cap (\eta + \mathbb{Z}_+\alpha)$ for all $c \in \mathbb{C}^\times$. \square

Theorem 4.9 *Let F be a face. Then*

- (a) $N_{U_\alpha}^\subseteq(V_F) = N_{U_\alpha}(V_F) = \begin{cases} U_\alpha, & \text{if } \alpha \in \Delta_p(F) \cup \Delta(F), \\ \{1\}, & \text{otherwise.} \end{cases}$
- (b) $Z_{U_\alpha}(V_F) = \begin{cases} U_\alpha, & \text{if } \alpha \in \Delta_p(F) \cup \Delta_*(F), \\ \{1\}, & \text{otherwise.} \end{cases}$

Proof. The theorem holds for $F = \emptyset$ by (25). Let $F \neq \emptyset$. We make repeated use of Lemma 4.8 in the proof without mentioning it further. Let $u \in U_\alpha$ and $v_\eta \in V_\eta$ with $\eta \in F$. If $\alpha \in \Delta_p(F) \cup \Delta_*(F)$ then

$$\text{supp}(uv_\eta) \subseteq P(V) \cap (\eta + \mathbb{Z}_+\alpha) = \{\eta\}$$

by Theorem 4.6 (a) and Theorem 4.4 (b). Therefore, $uv_\eta = v_\eta$. Hence, $U_\alpha \subseteq Z_{U_\alpha}(V_F) \subseteq N_{U_\alpha}(V_F) \subseteq N_{U_\alpha}^\subseteq(V_F) \subseteq U_\alpha$. If $\alpha \in \Delta^*(F)$ then

$$\text{supp}(uv_\eta) \subseteq P(V) \cap (\eta + \mathbb{Z}_+\alpha) \subseteq F$$

by Theorem 4.4 (a). Therefore, $uv_\eta \in V_F$. Hence, $N_{\bar{U}_\alpha}^\subseteq(V_F) = U_\alpha$. Since $N_{\bar{U}_\alpha}^\subseteq(V_F)$ is a group, we have $N_{U_\alpha}(V_F) = N_{\bar{U}_\alpha}^\subseteq(V_F)$.

Let $\alpha \in \Delta_n(F)$ and $u \in U_\alpha \setminus \{1\}$. It follows from Theorem 4.6 (b) that there exists an F -weight η such that $P(V) \cap (\eta + \mathbb{Z}_+\alpha) \not\subseteq F$. There exists $v_\eta \in V_\eta$ such that

$$\text{supp}(uv_\eta) = P(V) \cap (\eta + \mathbb{Z}_+\alpha) \not\subseteq F.$$

Hence $uv_\eta \notin V_F$. It follows that $\{1\} \subseteq Z_{U_\alpha}(V_F) \subseteq N_{U_\alpha}(V_F) \subseteq N_{\bar{U}_\alpha}^\subseteq(V_F) \subseteq \{1\}$. Let $\alpha \in \Delta^*(F)$ and $u \in U_\alpha \setminus \{1\}$. It follows from Theorem 4.4 (c) that there exists an F -weight η such that $P(V) \cap (\eta + \mathbb{Z}_+\alpha) \neq \{\eta\}$. There exists $v_\eta \in V_\eta$ such that

$$\text{supp}(uv_\eta) = P(V) \cap (\eta + \mathbb{Z}_+\alpha) \neq \{\eta\}.$$

Hence $uv_\eta \neq v_\eta$. We have shown that $Z_{U_\alpha}(V_F) = \{1\}$. \square

Theorem 4.10 *Let F be a fundamental face. Then $N_{\bar{T}}^\subseteq(V_F) = N_T(V_F) = T$. Moreover, if F is nonempty then*

$$Z_T(V_F) = \left\{ \prod_{j=1}^{2m-l} c_j^{-\langle \mu, \alpha_j^\vee \rangle} \rho(t_j(c_j)) \in T \mid \prod_{j=1}^{2m-l} c_j^{\langle \alpha_i, \alpha_j^\vee \rangle} = 1 \text{ for all } i \in \lambda^*(F) \right\},$$

and $Z_T(V_\emptyset) = T$. In particular, $T_{\lambda^*(F)} \subseteq Z_T(V_F)$.

Proof. The theorem holds for $F = \emptyset$ by (25). Let $F \neq \emptyset$. We have $N_{\bar{T}}^\subseteq(V_F) = T$ because V_F is a direct sum of T -invariant weight spaces. Since $N_{\bar{T}}^\subseteq(V_F)$ is a group we get $N_T(V_F) = N_{\bar{T}}^\subseteq(V_F)$.

Let $t = c \prod_{j=1}^{2m-l} \rho(t_j(c_j)) \in T$. If $\eta \in P(V)$ and $v_\eta \in V_\eta$ then $tv_\eta = c \prod_{j=1}^{2m-l} c_j^{\langle \eta, \alpha_j^\vee \rangle} v_\eta$. Therefore, $t \in Z_T(V_F)$ if and only if

$$c \prod_{j=1}^{2m-l} c_j^{\langle \eta, \alpha_j^\vee \rangle} = 1 \quad \text{for all } \eta \in P(V) \cap F. \quad (26)$$

Since $\mu \in P(V) \cap F$ it follows from (26) that

$$c \prod_{j=1}^{2m-l} c_j^{\langle \mu, \alpha_j^\vee \rangle} = 1. \quad (27)$$

Let $\alpha \in \lambda^*(F)$. By Theorem 4.4 (c) and (a) there exists an F -weight η such that $\langle \eta, \alpha^\vee \rangle < 0$, and the whole α -string through η is contained in $P(V) \cap F$. In particular, $\eta + \alpha \in P(V) \cap F$. Inserting into (26), we find

$$c \prod_{j=1}^{2m-l} c_j^{\langle \eta, \alpha_j^\vee \rangle} = 1 \quad \text{and} \quad c \prod_{j=1}^{2m-l} c_j^{\langle \eta + \alpha, \alpha_j^\vee \rangle} = 1.$$

We conclude that

$$\prod_{j=1}^{2m-l} c_j^{\langle \alpha, \alpha_j^\vee \rangle} = 1 \quad \text{for all } \alpha \in \lambda^*(F). \quad (28)$$

Conversely, (27) and (28) imply (26) because we have $P(V) \cap F \subseteq \mu - \mathbb{R}^+ \lambda^*(F)$ by Theorem 3.3. We have shown that $t = c \prod_{j=1}^{2m-l} \rho(t_j(c_j)) \in Z_T(V_F)$ if and only if (27) and (28) hold. From

$$\begin{aligned} \lambda_*(F) &= \{j \in J_0 \mid \langle \alpha_i, \alpha_j^\vee \rangle = 0 \text{ for all } i \in \lambda^*(F)\} \\ &= \{j \in \mathbf{m} \mid \langle \mu, \alpha_j^\vee \rangle = 0 \text{ and } \langle \alpha_i, \alpha_j^\vee \rangle = 0 \text{ for all } i \in \lambda^*(F)\} \end{aligned}$$

we get $T_{\lambda_*(F)} \subseteq Z_T(V_F)$. □

Theorem 4.11 *If F is a fundamental face then*

$$N_N^\subseteq(V_F) = N_N(V_F) = TN_{\lambda(F)} \quad \text{and} \quad Z_N(V_F) = Z_T(V_F)N_{\lambda_*(F)}.$$

Proof. For $i \in \mathbf{m}$ we have

$$\rho(n_i) = \rho(\exp(e_i) \exp(-f_i) \exp(e_i)) \in U_{\alpha_i} U_{-\alpha_i} U_{\alpha_i}.$$

From Theorem 4.9 it follows that $\rho(n_i) \in N_N(V_F)$ for all $i \in \lambda(F)$, and $\rho(n_i) \in Z_N(V_F)$ for all $i \in \lambda_*(F)$. Hence

$$TN_{\lambda(F)} = N_T(V_F)N_{\lambda(F)} \subseteq N_N(V_F) \subseteq N_N^\subseteq(V_F) \quad \text{and} \quad Z_T(V_F)N_{\lambda_*(F)} \subseteq Z_N(V_F).$$

Let $n_w \in N_N^\subseteq(V_F)$ project to $w \in W$. From $n_w V_F \subseteq V_F$ and (3) we find $w(P(V) \cap F) \subseteq F$. In the proof of Theorem 4.3 we have seen that $F = \text{co}(P(V) \cap F)$. It follows that $wF = \text{co}(w(P(V) \cap F)) \subseteq F$. Since wF and F are faces of H of the same dimension we obtain $wF = F$. Hence, $w \in W(F)$ and $n_w \in TN_{\lambda(F)}$.

Let $n_w \in Z_N(V_F)$ project to $w \in W$. From $n_w v_\eta = v_\eta$ for all $\eta \in P(V) \cap F$ and (3) we get $w\eta = \eta$ for all $\eta \in P(V) \cap F$. Hence, $w \in W_{\lambda_*(F)}$ by Theorem 4.3 (b), and we can write $n_w = tn$ with $t \in T$ and $n \in N_{\lambda_*(F)} \subseteq Z_N(V_F)$. We conclude that $t = n_w n^{-1} \in Z_N(V_F) \cap T = Z_T(V_F)$. Thus, $n_w \in Z_T(V_F)N_{\lambda_*(F)}$. □

Theorem 4.12 *If F is a fundamental face then*

$$N_G^\subseteq(V_F) = N_G(V_F) = P_{\lambda(F)} \quad \text{and} \quad Z_G(V_F) = G_{\lambda_*(F)} Z_T(V_F) \ltimes U^{\lambda(F)}.$$

Proof. From Theorem 4.9 (a) it follows that $U \subseteq N_G(V_F)$. Let $g = un u_1$ where $n \in N$ and $u, u_1 \in U$. From Theorem 4.11 we observe

$$gV_F \subseteq V_F \Leftrightarrow nu_1 V_F \subseteq u^{-1} V_F \Leftrightarrow nV_F \subseteq V_F \Leftrightarrow n \in TN_{\lambda(F)}.$$

Thus $N_G^\subseteq(V_F) = UTN_{\lambda(F)}U = P_{\lambda(F)}$. Since $N_G^\subseteq(V_F)$ is a group, we have $N_G(V_F) = N_G^\subseteq(V_F)$.

The group $G_{\lambda_*(F)}$ is generated by the root groups U_α , $\alpha \in \Delta_*(F)$, which stabilize V_F pointwise by Theorem 4.9 (b). Hence $G_{\lambda_*(F)} \subseteq Z_G(V_F)$. From Theorem 4.9 (b) we obtain that $U_\alpha \subseteq Z_G(V_F)$ for all $\alpha \in \Delta_+^{re} \setminus \Delta(F)$. Moreover, $Z_G(V_F)$ is a normal subgroup of $N_G(V_F)$, and the group $N_G(V_F)$ contains U . Since $U^{\lambda(F)}$ is the normal subgroup of U

generated by U_α , $\alpha \in \Delta_+^{re} \setminus \Delta(F)$, we have $U^{\lambda(F)} \subseteq Z_G(V_F)$. Clearly, $Z_T(V_F) \subseteq Z_G(V_F)$. So, $G_{\lambda^*(F)} Z_T(V_F) U^{\lambda(F)} \subseteq Z_G(V_F)$.

Now let $g \in Z_G(V_F) \subseteq N_G(V_F) = P_{\lambda(F)}$. Since $P_{\lambda(F)} = TG_{\lambda^*(F)} G_{\lambda^*(F)} U^{\lambda(F)}$ we can write g as a product $g = xh$ for some $x \in TG_{\lambda^*(F)}$ and $h \in G_{\lambda^*(F)} U^{\lambda(F)} \subseteq Z_G(V_F)$. Because of the Bruhat decomposition $TG_{\lambda^*(F)} = U_{\lambda^*(F)} TN_{\lambda^*(F)} U_{\lambda^*(F)}$ we can write $x = u_1 n u$ with $u_1, u \in U_{\lambda^*(F)}$ and $n \in TN_{\lambda^*(F)}$. Since $x = gh^{-1} \in Z_G(V_F)$ and $u^{-1}V_F = V_F$ we find

$$u_1 n v_\eta = u_1 n u u^{-1} v_\eta = x u^{-1} v_\eta = u^{-1} v_\eta \quad \text{for all } v_\eta \in V_\eta, \eta \in P(V) \cap F.$$

Comparing the components of smallest weight, we obtain $n v_\eta = v_\eta$ for all $v_\eta \in V_\eta$, $\eta \in P(V) \cap F$. Hence, by Theorem 4.11,

$$n \in Z_N(V_F) \cap TN_{\lambda^*(F)} = Z_T(V_F) N_{\lambda^*(F)} \cap TN_{\lambda^*(F)} \subseteq Z_T(V_F) T_{\lambda^*(F)} = Z_T(V_F).$$

It follows that $x \in Z_T(V_F) U_{\lambda^*(F)}$. With $TG_{\lambda^*(F)} = U_{\lambda^*(F)}^- TN_{\lambda^*(F)} U_{\lambda^*(F)}^-$ we get similarly $x \in Z_T(V_F) U_{\lambda^*(F)}^-$. We conclude that $x \in Z_T(V_F) U_{\lambda^*(F)} \cap Z_T(V_F) U_{\lambda^*(F)}^- = Z_T(V_F)$ and $g = xh \in Z_T(V_F) G_{\lambda^*(F)} U^{\lambda(F)}$. \square

The left and right centralizers of $e \in E(M)$ in G are defined by

$$C_G^l(e) := \{g \in G \mid ge = ege\} \quad \text{and} \quad C_G^r(e) := \{g \in G \mid eg = ege\}.$$

The centralizer of $e \in E(M)$ in G is defined by

$$C_G(e) := \{g \in G \mid ge = eg\} = C_G^l(e) \cap C_G^r(e).$$

From Proposition 4.7 (a), (b), (c) and Theorem 4.12, we obtain the following corollary, which is one of the main results of this section.

Corollary 4.13 *Let F be a fundamental face. Then:*

- (a) $C_G^l(e(F)) = P_{\lambda(F)}$ and $C_G^r(e(F)) = P_{\lambda(F)}^-$.
- (b) $C_G(e(F)) = L_{\lambda(F)}$.

The left and right stabilizers of $e \in E(M)$ in G are defined by

$$S_G^l(e) := \{g \in G \mid ge = e\} \quad \text{and} \quad S_G^r(e) := \{g \in G \mid eg = e\}.$$

The stabilizer of $e \in E(M)$ in G is defined by

$$S_G(e) := \{g \in G \mid ge = eg = e\} = S_G^l(e) \cap S_G^r(e).$$

Combining Proposition 4.7 (d), (e) and Theorem 4.12, we get the corollary below, which is another main result of this section. Note also that the group $Z_T(V_F)$ has been described explicitly in Theorem 4.10.

Corollary 4.14 *Let F be a fundamental face. Then:*

- (a) $S_G^l(e(F)) = G_{\lambda_*(F)} Z_T(V_F) \ltimes U^{\lambda(F)}$ and $S_G^r(e(F)) = U_-^{\lambda(F)} \rtimes G_{\lambda_*(F)} Z_T(V_F)$.
- (b) $S_G(e(F)) = G_{\lambda_*(F)} Z_T(V_F)$.

As a consequence of Proposition 4.7 and Theorem 4.9 we get the following properties of the root groups, which we often use.

Corollary 4.15 *Let F be a face and set $e = e(F)$. Let $\alpha \in \Delta^{re}$ and $u \in U_\alpha \setminus \{1\}$.*

- (a) *If $\alpha \in \Delta_*(F)$ then $ue = eu = e$.*
- (b) *If $\alpha \in \Delta^*(F)$ then $ue = eu \neq e$.*
- (c) *If $\alpha \in \Delta_p(F)$ then $ue = e \neq eu$.*
- (d) *If $\alpha \in \Delta_n(F)$ then $eu = e \neq ue$.*

We employ many times the following consequence of Corollaries 4.13 and 4.14.

Corollary 4.16 *Let F be a fundamental face.*

- (a) *Write $u \in U$ as a product $u = u_1 u_2$ with $u_1 \in U_{\lambda^*(F)}$ and $u_2 \in U_{\lambda_*(F)} \ltimes U^{\lambda(F)}$. Then*

$$ue(F) = u_1 e(F) = e(F) u_1.$$

- (b) *Write $u \in U^-$ as a product $u = u_1 u_2$ with $u_1 \in U_-^{\lambda(F)} \rtimes U_{\lambda^*(F)}^-$ and $u_2 \in U_{\lambda^*(F)}^-$. Then*

$$e(F)u = e(F)u_2 = u_2 e(F).$$

Let F be a fundamental face. We denote the projections that belong to the semidirect product decompositions $P_{\lambda(F)} = L_{\lambda(F)} \ltimes U^{\lambda(F)}$ and $P_{\lambda(F)}^- = U_-^{\lambda(F)} \rtimes L_{\lambda(F)}$ by

$$\theta_F : P_{\lambda(F)} \rightarrow L_{\lambda(F)} \quad \text{and} \quad \theta_F^- : P_{\lambda(F)}^- \rightarrow L_{\lambda(F)}.$$

Note that θ_F and θ_F^- are morphisms of groups whose restrictions to $L_{\lambda(F)} = P_{\lambda(F)} \cap P_{\lambda(F)}^-$ coincide.

Proposition 4.17 *Let F be a fundamental face. If $g \in P_{\lambda(F)}$ and $h \in P_{\lambda(F)}^-$, then*

- (a) $ge(F) = \theta_F(g)e(F) = e(F)\theta_F(g)$.
- (b) $e(F)h = e(F)\theta_F^-(h) = \theta_F^-(h)e(F)$.

Proof. Let $g = \theta_F(g)g_1$ where $\theta_F(g) \in L_{\lambda(F)}$ and $g_1 \in U^{\lambda(F)}$. From Corollaries 4.13 and 4.14 we find that

$$ge(F) = \theta_F(g)g_1 e(F) = \theta_F(g)e(F) = e(F)\theta_F(g),$$

which shows (a). Applying the Chevalley anti-involution we obtain (b). \square

In Theorem 4.22 below we show that the elements of M can be written in the form

$$ae(F)b \quad \text{where} \quad a, b \in G, \quad F \in \mathcal{F}.$$

The following theorem is another main result of this section. It describes the equality of such expressions. In combination with the Birkhoff decomposition of G and Lemma 4.1 the theorem allows to describe their multiplication.

Theorem 4.18 *Let $g, h \in G$ and F, F' be fundamental faces. The following are equivalent.*

- (a) $ge(F) = e(F')h$.
- (b) $F = F'$, $g \in P_{\lambda(F)}$, $h \in P_{\lambda(F)}^-$, and $\theta_F(g) = \theta_F^-(h)g_1$ for some $g_1 \in G_{\lambda_*(F)}Z_T(V_F)$.

Proof. We first show that (b) implies (a). Since $F' = F$, from Proposition 4.17 (a), Corollary 4.14, and Proposition 4.17 (b) we get

$$ge(F) = \theta_F(g)e(F) = \theta_F^-(h)g_1e(F) = \theta_F^-(h)e(F) = e(F)h.$$

Next, we prove that (a) implies (b). Comparing the images of $ge(F)$ and $e(F')h$ we find $gV_F = V_{F'}$. Let $g = unv$ where $u, v \in U$, and $n \in N$ projecting to $w \in W$. From Theorem 4.12 we obtain $uV_F = V_F$ and $vV_{F'} = V_{F'}$. So, $nV_F = V_{F'}$. Hence, $wF = F'$, where F and F' are fundamental faces. It follows that $F = F'$.

We observe from $ge(F) = e(F)h$ that $gV_F = V_F$ and $hV_F^\perp = V_F^\perp$. Then $g \in N_G(V_F) = P_{\lambda(F)}$ and $h \in N_G(V_F^\perp) = N_{G^*}(V_F)^* = N_G(V_F)^* = P_{\lambda(F)}^-$ by Theorem 4.12 and (24). Thanks to Proposition 4.17, we obtain

$$\theta_F(g)e(F) = ge(F) = e(F)h = \theta_F^-(h)e(F).$$

From Corollary 4.14 it follows that

$$\theta_F^-(h)^{-1}\theta_F(g) \in (G_{\lambda_*(F)}Z_T(V_F) \ltimes U^{\lambda(F)}) \cap L_{\lambda(F)} = G_{\lambda_*(F)}Z_T(V_F).$$

Thus $\theta_F(g) = \theta_F^-(h)g_1$ for some $g_1 \in G_{\lambda_*(F)}Z_T(V_F)$. □

From the preceding theorem and Lemma 4.1 we obtain easily the following result.

Corollary 4.19 *Let F and F' be faces of H . If $Ge(F)G = Ge(F')G$, then there exists $w \in W$ such that $F' = wF$.*

4.4 The cross-section lattice

The *cross-section lattice* of M relative to T and B is defined by

$$\Lambda := \{e(F) \mid F \in \mathcal{F}\} \subseteq E(\overline{T}). \quad (29)$$

It is a finite monoid isomorphic to (\mathcal{F}, \cap) . Its elements are called *fundamental idempotents*.

To obtain an alternative description of Λ we need the following lemma. Its proof is straightforward.

Lemma 4.20 *Let $e \in E(\overline{T})$. Then the following conditions are equivalent.*

- (a) $Be = eBe$.
- (b) $Be \subseteq eB$.
- (c) $be = ebe$ for all $b \in B$.
- (d) $ue = eue$ for all $u \in U$.
- (e) $Ue \subseteq eU$.
- (f) $Ue = eUe$.

The cross-section lattice Λ can be obtained by the Borel subgroup B as follows.

Theorem 4.21 *We have $\Lambda = \{e \in E(\overline{T}) \mid Be = eBe\}$.*

Proof. For any $e \in \Lambda$, it follows from Corollary 4.16 and Lemma 4.20 that $Be = eBe$.

Now let $e \in E(\overline{T}) \setminus \Lambda$. Then $e = e(wF')$ for some fundamental face F' and $w \in W \setminus W(F')$. Since $w^{-1} \notin W(F')$ there exists a root $\alpha \in \Delta_+^{re} \setminus \Delta(F')$ such that $w^{-1}\alpha \in \Delta_-^{re} \setminus \Delta(F')$. Let $u \in U_\alpha \setminus \{1\}$. From (d) of Corollary 4.15 we have $eu = e \neq ue$, hence $eue = e \neq ue$. Therefore, $Be \neq eBe$ by Lemma 4.20. \square

The cross-section lattice Λ and the $G \times G$ -orbit decomposition of M are related.

Theorem 4.22

$$M = G\Lambda G = \bigsqcup_{e \in \Lambda} GeG.$$

Proof. From Lemma 4.1 it follows that every element of M can be written in the form

$$ege_1g_1 \cdots e_kg_k$$

where $g, g_i \in G$ and $e, e_i \in \Lambda$ for $i = 1, \dots, k$. We now show how this element can be further reduced to an element of $G\Lambda G$. It suffices to show that $e(F)ge(F_1) \in G\Lambda G$ for all $g \in G$ and $F, F_1 \in \mathcal{F}$. Since $G = B^-NB$, we see that $g = vnu$ for some $v \in U^-$, $n \in N$, and $u \in U$. Let $v = v_1v_2$ for some $v_1 \in U_-^{\lambda(F)}U_{\lambda^*(F)}^-$ and $v_2 \in U_{\lambda^*(F)}^-$, and let $u = u_1u_2$ for some $u_1 \in U_{\lambda^*(F_1)}$ and $u_2 \in U_{\lambda^*(F_1)}U^{\lambda(F_1)}$. From Corollary 4.16 it follows that

$$e(F)ge(F_1) = e(F)vnu e(F_1) = v_2e(F)ne(F_1)u_1 = v_2e(F)(ne(F_1)n^{-1})nu_1.$$

Here $e(F)(ne(F_1)n^{-1}) \in E(\overline{T})$ by Lemma 4.1 and (23). Hence,

$$e(F)(ne(F_1)n^{-1}) = n_2e(F_2)n_2^{-1}$$

for some $n_2 \in N$, $F_2 \in \mathcal{F}$. Thanks to Theorem 4.18, the decomposition is disjoint. \square

4.5 The unit regularity of $M(\rho)$

The following Lemma is used to determine the idempotents of M .

Lemma 4.23 *Let $g \in G$ and let F be a fundamental face. Then $e(F)ge(F) = e(F)$ if and only if g has a product decomposition of the form*

$$g = u_-xu_+ \quad \text{with} \quad u_- \in U_-^{\lambda(F)}, x \in G_{\lambda^*(F)}Z_T(V_F), u_+ \in U^{\lambda(F)}. \quad (30)$$

Moreover, this decomposition is unique, and

$$e(F)u_- = e(F) \quad \text{and} \quad xe(F) = e(F)x = e(F) \quad \text{and} \quad u_+e(F) = e(F). \quad (31)$$

Proof. Suppose that $g \in G$ has a decomposition $g = u_-xu_+$ of the form (30). By Corollary 4.14 we obtain the equations (31), from which we get $e(F)ge(F) = e(F)$ immediately.

If $g = u'_-x'u'_+$ is another decomposition of the form (30) then

$$G_{\lambda_*(F)}Z_T(V_F) \rtimes U^{\lambda(F)} \ni xu_+(u'_+)^{-1} = (u_-)^{-1}u'_-x' \in U_-^{\lambda(F)} \rtimes G_{\lambda_*(F)}Z_T(V_F) \quad (32)$$

Since $P_{\lambda(F)} \cap P_{\lambda(F)}^- = L_{\lambda(F)}$ we get $xu_+(u'_+)^{-1} = (u_-)^{-1}u'_-x' \in G_{\lambda_*(F)}Z_T(V_F)$. We conclude that $u_+(u'_+)^{-1}, (u_-)^{-1}u'_- \in G_{\lambda_*(F)}Z_T(V_F)$. From the semidirect decompositions used in (32) it follows that $u_+(u'_+)^{-1} = 1$ and $(u_-)^{-1}u'_- = 1$. Hence, $x = x'$.

Now let $g \in G$ such that $e(F)ge(F) = e(F)$. We may write g in the form $g = u_1u_2nv_1v_2$ where $u_1 \in U_-^{\lambda(F)}$, $u_2 \in U_{\lambda(F)}^-$, $v_1 \in U_{\lambda(F)}$, $v_2 \in U^{\lambda(F)}$, and $n \in N$ projecting to $w \in W$. From Proposition 4.17 it follows that

$$e(F) = e(F)ge(F) = u_2e(F)ne(F)v_1 = u_2e(F)(ne(F)n^{-1})nv_1.$$

By Lemma 4.1 and (23) we find that $e(F)(ne(F)n^{-1}) = e(F)e(wF) = e(F \cap wF)$. Hence, $e(F) = u_2e(F \cap wF)nv_1$. From Corollary 4.19 we find that the dimension of the faces F , wF , and $F \cap wF$ are the same. Since $F \cap wF$ is contained in F and wF we obtain $F = F \cap wF = wF$. Thus, $w \in W_{\lambda(F)}$ and $u_2^{-1}e(F) = e(F)nv_1$. From Theorem 4.18 it follows that $u_2^{-1} = nv_1g_1$ for some $g_1 \in G_{\lambda_*(F)}Z_T(V_F)$. Therefore, $g = u_1u_2nv_1v_2 = u_1g_1^{-1}v_2$. \square

Theorem 4.24 (a) *The unit group of M is G .*

(b) *The set of idempotents of M is*

$$E(M) = \{geg^{-1} \mid e \in E(\overline{T}), g \in G\} = \{geg^{-1} \mid e \in \Lambda, g \in G\}.$$

(c) $M = GE(M) = E(M)G$. *Hence, the monoid M is unit regular.*

Proof. We make use of Theorem 4.22 in the proof without mentioning it further.

Clearly, G is a subgroup of the unit group of M . Now let $ge(F)h$ be a unit of M , where $g, h \in G$ and F is a fundamental face. Then

$$e(F) = g^{-1}(ge(F)h)h^{-1}$$

is a unit. Thus $e(F) = 1$, and hence $ge(F)h = gh \in G$.

It is straightforward that $\{geg^{-1} \mid e \in \Lambda, g \in G\} \subseteq \{geg^{-1} \mid e \in E(\overline{T}), g \in G\} \subseteq E(M)$. Now let $ge(F)h$ be an idempotent of M , where $g, h \in G$ and F is a fundamental face. Then $e(F) = e(F)hge(F)$. Let $hg = u_-xu_+$ be a decomposition as in Lemma 4.23. By Lemma 4.23 (31) we find

$$ge(F)h = ge(F)hgg^{-1} = gu_+^{-1}u_+e(F)u_-xu_+g^{-1} = gu_+^{-1}e(F)(gu_+^{-1})^{-1}.$$

If $x \in M$ then $x = geh$ for some $e \in \Lambda$ and $g, h \in G$. So $x = gh(h^{-1}eh) = (geg^{-1})gh$. Hence, $M = GE(M) = E(M)G$. \square

Corollary 4.25 *If two idempotents of M are in the same $G \times G$ -orbit, then they are G -conjugate.*

Proof. Let e, e_1 be two idempotents of M in the same $G \times G$ -orbit. Then $e = ge'g^{-1}$ and $e_1 = g_1e'_1g_1^{-1}$ for some $g, g_1 \in G$ and $e', e'_1 \in \Lambda$. From Theorem 4.22 it follows that $e' = e'_1$. Hence, e and e_1 are G -conjugate. \square

4.6 The Renner monoid

We obtain a congruence relation on \overline{N} by

$$x_1 \sim x_2 \quad \text{if and only if} \quad x_1 T = x_2 T,$$

where $x_1, x_2 \in \overline{N}$. The quotient monoid

$$R := \overline{N} / \sim = \{xT \mid x \in \overline{N}\}$$

is called the Renner monoid of M relative to T . We denote by $\varphi : \overline{N} \rightarrow R$ the quotient morphism: $\varphi(n) := nT$, $n \in N$.

For $Y \subseteq R$ we denote by $E(Y)$ the set of idempotents of R contained in Y . Clearly, if Y is a submonoid of R then $E(Y)$ is the set of idempotents of Y .

Theorem 4.26 (a) *The unit group of R is W .*

(b) *The quotient morphism restricts to a bijective map from $E(\overline{N})$ to $E(R)$. In particular, $E(R)$ is a commutative submonoid of R .*

(c) *$R = WE(R) = E(R)W$. Hence, the monoid R is unit regular.*

Proof. We denote the unit group of R by R^\times . Applying the quotient morphism to $\overline{N} = NE(\overline{N}) = E(\overline{N})N$, we obtain

$$R = W\phi(E(\overline{N})) = \phi(E(\overline{N}))W,$$

where $W = \phi(N) \subseteq R^\times$ and $\phi(E(\overline{N})) \subseteq E(R)$. Moreover, $\phi(E(\overline{N}))$ is a commutative submonoid of R , since $E(\overline{N})$ is a commutative submonoid of \overline{N} .

Let $x \in R^\times$. Then $x = we$ for some $w \in W$ and $e \in \phi(E(\overline{N})) \subseteq E(R)$. We conclude that $e = w^{-1}x \in E(R) \cap R^\times = \{1\}$. Thus $x = w \in W$.

Let $x \in E(R)$. Then $x = \phi(ne(F))$ for some $n \in N$ and $F \in \mathcal{F}(H)$. We set $w = \phi(n) \in W$. Since $x = x^2$, it follows from Lemma 4.1 and (23) that there exists $t \in T$ such that

$$e(F)t = e(F)ne(F) = e(F)ne(F)n^{-1}n = e(F \cap wF)n. \quad (33)$$

From Proposition 4.7 (e) we obtain $F = F \cap wF$, which is equivalent to $F \subseteq wF$. Since F and wF are faces of the same dimension we get $wF = F$. By Lemma 4.1 and (33) we find

$$ne(F) = e(wF)n = e(F)n = e(F)t,$$

which shows that $x = \phi(ne(F)) = \phi(e(F)t) = \phi(e(F)) \in \phi(E(\overline{N}))$.

Let $\phi(e(F)) = \phi(e(F_1))$ with $F, F_1 \in \mathcal{F}(H)$. Then there exists $t \in T$ such that $e(F)t = e(F_1)$. By Proposition 4.7 (e) we get $F = F_1$. Hence $e(F) = e(F_1)$. \square

We identify $E(\overline{N})$ and $E(R)$ by the quotient morphism, which is possible by Theorem 4.26 (b). This is common in the theory of reductive linear algebraic monoids. In particular, we write $e(F)$ instead of $e(F)T$ for all $F \in \mathcal{F}(H)$. If it is not clear from the context to which space the idempotent $e(F)$, $F \in \mathcal{F}(H)$, belongs, we write explicitly $e(F) \in \overline{N}$ or $e(F) \in R$.

We denote the image of the cross-section lattice Λ defined in (29) under the quotient morphism still by Λ . Thus, $\Lambda = \{e(F) \mid F \in \mathcal{F}\}$ in \overline{N} , as well as in R .

Corollary 4.27 *The monoid R is an inverse monoid. Its inverse map $^{\text{inv}} : R \rightarrow R$ is the anti-involution whose restriction to W is the inverse map, and to $E(R)$ is the identity map.*

Proof. From [9, Theorem 5.1.1] and Theorem 4.26 (b), (c) it follows that R is an inverse monoid. By Theorem 4.26 (c) every element of R is of the form σe with $\sigma \in W$ and $e \in E(R)$. Now, $(\sigma e)(e\sigma^{-1})(\sigma e) = \sigma e$ and $(e\sigma^{-1})(\sigma e)(e\sigma^{-1}) = e\sigma^{-1}$. Hence $(\sigma e)^{\text{inv}} = e\sigma^{-1}$. \square

Remark 4.28 *The restriction of the Chevalley anti-involution to $\overline{N} = \overline{N}^*$ induces the inverse map on R : $\phi(n)^{\text{inv}} = \phi(n^*)$ for all $n \in \overline{N}$.*

We have seen in Theorem 4.26 (b) that the set of idempotents $E(R)$ is a commutative submonoid of R . By [9, Proposition 1.3.2] we obtain a partial order on $E(R)$ by

$$e \leq f :\Leftrightarrow ef = e, \quad e, f \in E(R),$$

and $(E(R), \leq)$ is a lower semilattice. Furthermore, the multiplication of $E(R)$ coincides with the semilattice intersection:

$$ef = e \wedge f, \quad e, f \in E(R).$$

The Weyl group W acts on $E(R)$ by conjugation, preserving the partial order.

Corollary 4.29 *The map*

$$\begin{aligned} \mathcal{F}(H) &\rightarrow E(R) \\ F &\mapsto e(F) \end{aligned}$$

has the following properties:

- (a) *It is a W -equivariant isomorphism of the partially ordered sets $(\mathcal{F}(H), \subseteq)$ and $(E(R), \leq)$. In particular, $(E(R), \leq)$ is a lattice.*
- (b) *It is a W -equivariant isomorphism of the monoids $(\mathcal{F}(H), \cap)$ and $E(R)$.*

Proof. We first show (b). The W -equivariance of the map follows by applying the quotient morphism ϕ to the equation in Lemma 4.1. The idempotents $e(F) \in \overline{N}$, $F \in \mathcal{F}(H)$, are pairwise different. Hence the map is bijective by Proposition 4.2. We conclude from (23) that it is also a monoid homomorphism.

For $F, F_1 \in \mathcal{F}(H)$ we have $F \subseteq F_1$ if and only if $F = F \cap F_1$. Therefore, we obtain (a) from (b) and the definition of the partial order on $E(R)$. \square

The following lemma refines Theorem 4.26 (c).

Lemma 4.30

$$R = \bigsqcup_{e \in E(R)} We = \bigsqcup_{e \in E(R)} eW,$$

and $E(We) = E(eW) = \{e\}$ for all $e \in E(R)$.

Proof. We have $R = \bigcup_{e \in E(R)} eW$ by Theorem 4.26 (c). To show that this union is disjoint let $e, e_1 \in E(R)$ such that $eW \cap e_1W \neq \emptyset$. Then $e_1 = ew$ for some $w \in W$. We observe that $ee_1 = ew = e_1$. From $e = e_1w^{-1}$ we get $e_1e = e$. But e and e_1 commute by Theorem 4.26 (b), so $e = e_1$. It also follows that $E(eW) = \{e\}$.

The remaining statements $R = \bigsqcup_{e \in E(R)} We$, and $E(We) = \{e\}$ for all $e \in E(R)$ follow similarly. \square

For $x \in R$ we denote by $Cl_W(x) := \{wxw^{-1} \mid w \in W\}$ the set of W -conjugates of x . From Corollary 4.29 and Corollary 3.17 we obtain

$$E(R) = \bigcup_{w \in W} w\Lambda w^{-1} = \bigsqcup_{e \in \Lambda} Cl_W(e). \quad (34)$$

Lemma 4.31 *The Renner monoid R is generated by S and Λ . We have*

$$R = W\Lambda W = \bigsqcup_{e \in \Lambda} WeW,$$

and $E(WeW) = Cl_W(e)$ for all $e \in \Lambda$.

Proof. Let $e \in \Lambda$. Clearly, $Cl_W(e) \subseteq E(WeW)$. Conversely, for any idempotent $e' \in E(WeW)$, there exist $w, w_1 \in W$ such that $e' = wew_1 = (wew^{-1})ww_1$. Then $e' = wew^{-1}$ by Lemma 4.30.

Theorem 4.26 (c) and (34) show that $R = \bigcup_{e \in \Lambda} WeW$. So R is generated by S and Λ . To show that the union is disjoint let $e, f \in \Lambda$ such that $WeW \cap WfW \neq \emptyset$. Then $WeW = WfW$, from which we get $Cl_W(e) = Cl_W(f)$. Now $e = f$ follows from (34). \square

We continue to use the examples in Section 3 to describe their cross-section lattices and Renner monoids.

Example 4.1 Let H be the weight hull of μ as in Example 3.1, where $\mu = 3\mu_1 + 2\mu_2$ is a dominant weight of the Kac-Moody algebra \mathfrak{g} of type A_2 .

The set of idempotents $E(R)$ consists of 14 elements since H has 14 faces. The cross-section lattice

$$\Lambda = \{0, e_1, e_2, e_3, 1\}$$

has five idempotents, where e_1 is the idempotent determined by the vertex face $\{\mu\}$, e_2 is the idempotent corresponding to the face $\overline{\mu\tau_1\mu}$, and e_3 corresponds to the face $\overline{\mu\tau_2\mu}$. The Renner monoid is finite and

$$R = \{0\} \sqcup We_1W \sqcup We_2W \sqcup We_3W \sqcup W.$$

Example 4.2 Let H be the weight hull of μ as in Example 3.2, where $\mu = \mu_1$ is the first fundamental dominant weight of the affine Kac-Moody Lie algebra \mathfrak{g} of type $A_1^{(1)}$.

The set of idempotents $E(R)$ is infinite since H has infinitely many faces. The cross-section lattice

$$\Lambda = \{0, e_1, e_2, 1\}$$

has 4 elements, where e_1 is the idempotent determined by the vertex face $\{\mu\}$, and e_2 is the idempotent determined by the face $\overline{\mu\tau_1\mu}$. The Renner monoid is infinite and

$$R = \{0\} \sqcup We_1W \sqcup We_2W \sqcup W.$$

Example 4.3 Let H be the weight hull of μ as in Example 3.3, where $\mu = \mu_1 + \mu_2$ is a dominant weight of the indefinite, strongly hyperbolic Kac-Moody Lie algebra $\mathfrak{g}(A)$ with

$$A = \begin{pmatrix} 2 & -2 \\ -3 & 2 \end{pmatrix}.$$

The set of idempotents $E(R)$ is infinite since H has infinitely many faces. The cross-section lattice

$$\Lambda = \{0, e_1, e_2, e_3, 1\}$$

has 5 elements, where e_1 is the idempotent determined by the vertex face $\{\mu\}$, and e_2 is the idempotent corresponding to the face $\overline{\mu\tau_1\mu}$, and e_3 corresponds to the face $\overline{\mu\tau_2\mu}$. The Renner monoid is infinite and

$$R = \{0\} \sqcup We_1W \sqcup We_2W \sqcup We_3W \sqcup W.$$

Let $e \in E(R)$. The left and right centralizers of e in W are defined by

$$C_W^l(e) := \{w \in W \mid we = ewe\} \quad \text{and} \quad C_W^r(e) := \{w \in W \mid ew = ewe\}.$$

The centralizer of e in W is defined by $C_W(e) := \{w \in W \mid we = ew\} = C_W^l(e) \cap C_W^r(e)$. Similarly, the left and right stabilizers of e in W are defined by

$$S_W^l(e) := \{w \in W \mid we = e\} \quad \text{and} \quad S_W^r(e) := \{w \in W \mid ew = e\}.$$

The stabilizer of e in W is defined by $S_W(e) := \{w \in W \mid we = ew = e\} = S_W^l(e) \cap S_W^r(e)$.

Theorem 4.32 *Let $e = e(F)$ where F is a face. Then:*

- (a) $C_W^l(e) = C_W^r(e) = C_W(e) = W(F)$.
- (b) $S_W^l(e) = S_W^r(e) = S_W(e) = W_*(F)$.

Proof. Applying the inverse map of R described in Corollary 4.27 to the equations which define the centralizers and stabilizers we find $C_W^r(e) = C_W^l(e)^{-1}$ and $S_W^r(e) = S_W^l(e)^{-1}$.

Because of $C_W^r(e) = C_W^l(e)^{-1}$ and $C_W(e) = C_W^r(e) \cap C_W^l(e)$ it is sufficient to show $C_W^l(e) = W(F)$ in (a). For $w \in W$ we find by Corollary 4.29 that

$$we(F) = e(F)we(F) = ww^{-1}e(F)we(F) = we(w^{-1}F \cap F)$$

if and only if $F = w^{-1}F \cap F$, if and only if $F \subseteq w^{-1}F$. Since F and $w^{-1}F$ are faces of the same dimension, $F \subseteq w^{-1}F$ is equivalent to $F = w^{-1}F$, which in turn is equivalent to $w \in W(F)$.

Since $S_W^r(e) = S_W^l(e)^{-1}$ and $S_W(e) = S_W^r(e) \cap S_W^l(e)$ it is sufficient to show $S_W^l(e) = W_*(F)$ in (b). It is sufficient to show this for a fundamental face F .

Let $w \in W$. Then $we(F) = e(F)$ in W if and only if there exists $n_w \in N$ projecting to w such that $n_w e(F) = e(F)$ in G . By Proposition 4.7 (d) and Theorem 4.11 this is equivalent to $w \in W_{\lambda_*(F)}$. From Theorem 3.15 we have $W_{\lambda_*(F)} = W_*(F)$. \square

As a supplement we now rewrite some of the results of this section in a form encountered in the theory of J -irreducible reductive linear algebraic monoids.

We define the *type map* $\lambda : \Lambda \rightarrow 2^\Pi$, and the maps $\lambda_* : \Lambda \rightarrow 2^\Pi$ and $\lambda^* : \Lambda \rightarrow 2^\Pi$ as follows: If $e = e(F)$ where F is a fundamental face of H then

$$\lambda(e) := \lambda(F) \quad \text{and} \quad \lambda_*(e) := \lambda_*(F) \quad \text{and} \quad \lambda^*(e) := \lambda^*(F).$$

From Theorems 3.15 and 4.32 we obtain the following characterizations. In the theory of reductive linear algebraic monoids these are often used as definitions, where the cross-section lattice Λ is obtained as in Theorem 4.21.

Corollary 4.33 *Let $e \in \Lambda$. Then we have*

$$\begin{aligned} \lambda(e) &= \{\alpha \in \Pi \mid r_\alpha e = er_\alpha\}, \\ \lambda_*(e) &= \{\alpha \in \Pi \mid r_\alpha e = er_\alpha = e\}, \\ \lambda^*(e) &= \{\alpha \in \Pi \mid r_\alpha e = er_\alpha \neq e\}. \end{aligned}$$

Corollary 4.33 can be used to eliminate in previous results the dependence on the faces of the orbit hull H . An example: For $e \in E(R)$ we set, as in the theory of reductive linear algebraic monoids, $W(e) := C_W(e)$ and $W_*(e) := S_W(e)$. From Theorems 3.15 and 4.32 we get the following corollary.

Corollary 4.34 *Let $e \in \Lambda$. Then*

$$W(e) = W_{\lambda(e)} = W_{\lambda_*(e)} \times W_{\lambda^*(e)} \quad \text{and} \quad W_*(e) = W_{\lambda^*(e)}.$$

Furthermore, $eW(e) = eW(e)e = W(e)e$ is a group with unit e , isomorphic to $W_{\lambda^*(e)}$.

The next corollary is a consequence of Corollary 3.14 and Theorem 3.15. The results are similar to those for J -irreducible reductive linear algebraic monoids obtained by M. S. Putcha and L. E. Renner in Corollary 4.12, Theorem 4.16, and Corollary 4.11 of [35].

Corollary 4.35 *The map $\lambda^* : \Lambda \rightarrow 2^\Pi$ restricts to an isomorphism of partially ordered sets*

$$\lambda^* : \Lambda \setminus \{0\} \rightarrow \{I \subseteq \Pi \mid I \text{ is } \mu\text{-connected}\}.$$

For $e \in \Lambda \setminus \{0\}$ we have $\lambda_*(e) = \{\alpha \in J_0 \setminus \lambda^*(e) \mid r_\alpha r_\beta = r_\beta r_\alpha \text{ for all } \beta \in \lambda^*(e)\}$.

Generalized Renner-Coxeter systems have been introduced by E. Godelle in [8, Definition 1.4] as a common concept for various sorts of monoids, called Renner monoids in the literature. Equivalent concepts can be found implicitly in the work of M. S. Putcha and L. E. Renner.

Theorem 4.36 *The triple (R, Λ, S) is a generalized Renner-Coxeter system, that is, the triple has the following properties:*

- (a) R is a unit regular monoid.
- (b) Denote by W the unit group of R . Then (W, S) is a Coxeter system.
- (c) Λ is a sub-semilattice of $E(R)$, and a cross-section for the action of W on $E(R)$.

- (d) For each pair $e_1 \leq e_2 \in E(R)$ there exist $w \in W$ and $f_1 \leq f_2 \in \Lambda$ such that $w^{-1}f_iw = e_i$ for $i = 1, 2$.
- (e) For each $e \in \Lambda$, both $W(e)$ and $W_*(e)$ are standard Coxeter subgroups of W .
- (f) The map $\Lambda \ni e \mapsto \lambda^*(e)$ is not decreasing: $e \leq f \Rightarrow \lambda^*(e) \subseteq \lambda^*(f)$.

Proof. Parts (a) and (b) hold by Theorem 4.26. Part (c) follows from Corollary 3.22 and (34). Part (e) holds by Corollary 4.34. Part (f) is obtained from Corollary 4.35 and $\lambda^*(0) = \emptyset$.

It remains to prove (d). There exist two faces $F_1 \subseteq F_2$ such that $e_1 = e(F_1)$ and $e_2 = e(F_2)$. By Corollary 3.17 there exists $w' \in W$ such that $w'F_2$ is a fundamental face. Clearly, $w'F_1$ is a face of $w'F_2$. It follows from Corollary 3.18 that there exists $w_1 \in W(w'F_2)$ such that $w_1w'F_1$ is a fundamental face contained in $w'F_2$. Moreover, $w_1w'F_1$ is face of $w_1w'F_2 = w'F_2$. Set $w := w_1w'$, $f_1 := e(wF_1)$, and $f_2 := e(wF_2)$. Then $f_1, f_2 \in \Lambda$ and $f_1 \leq f_2$. Furthermore, $w^{-1}f_iw = e_i$ for $i = 1, 2$. \square

4.7 The Bruhat and Birkhoff decompositions

The following decompositions of M are the first step to establish the Bruhat and Birkhoff decompositions.

Theorem 4.37 *Let $\epsilon \in \{+, -\}$. Then*

$$M = \bigsqcup_{e \in E(\overline{T})} GeB^\epsilon = \bigsqcup_{e \in E(\overline{T})} B^\epsilon eG.$$

Proof. Theorem 4.22 shows that

$$M = \bigsqcup_{f \in \Lambda} GfG. \quad (35)$$

Now let $f \in \Lambda$. From $G = \bigsqcup_{w \in W} B^-wB^\epsilon$ and Corollary 4.17 (b) we obtain

$$GfG = \bigcup_{w \in W} GfB^-wB^\epsilon = \bigcup_{w \in W} GfwB^\epsilon = \bigcup_{e \in Cl_W(f)} GeB^\epsilon.$$

This decomposition is also disjoint. Let $e = wf w^{-1}$ and $e_1 = w_1fw_1^{-1}$, and suppose that $GeB^\epsilon \cap Ge_1B^\epsilon \neq \emptyset$. Then there exist $g \in G$, $b \in B^\epsilon$, and $n, n_1 \in N$ projecting respectively to $w, w_1 \in W$ such that $gnfn^{-1}b = n_1fn_1^{-1}$. So

$$n^{-1}g^{-1}n_1f = fn^{-1}bn_1.$$

It follows from Theorem 4.18 that $n^{-1}bn_1 \in P_{\lambda(f)}^-$. Equivalently, $bn_1 \in wP_{\lambda(f)}^-$. But G is a disjoint union of $B^\epsilon xP_{\lambda(f)}^-$ where $x \in W/W_{\lambda(f)}$. Hence $w_1 \in wW_{\lambda(f)}$, so $e_1 = w_1fw_1^{-1} = wf w^{-1} = e$ by Corollary 4.34.

Inserting in (35), we get

$$M = \bigsqcup_{f \in \Lambda, e \in Cl_W(f)} GeB^\epsilon = \bigsqcup_{e \in E(\overline{T})} GeB^\epsilon,$$

and $M = \bigsqcup_{e \in E(\overline{T})} B^\epsilon eG$ is obtained by applying the Chevalley anti-involution. \square

We also need a technical Lemma.

Lemma 4.38 *Let F be a face of H and $w \in W(F)$. The following are equivalent:*

- (a) $w \in W_*(F)$.
- (b) $w\eta \geq \eta$ for all F -weights η .
- (c) $w\eta \leq \eta$ for all F -weights η .

Proof. Obviously, (a) implies (b) as well as (c). Now suppose that (b) holds. Let η' be an F -weight. Since $w \in W(F)$, the weights $w\eta', w^2\eta', w^3\eta', \dots$ are F -weights. By (b) we find

$$\eta' \leq w\eta' \leq w^2\eta' \leq w^3\eta' \leq \dots$$

All elements of this chain are smaller than or equal to μ . Since there are only finitely many weights of V between η' and μ this chain gets stationary. Thus, there exists $k \in \mathbb{Z}_+$ such that $w^{k+1}\eta' = w^k\eta'$. Hence $w\eta' = \eta'$. Now (a) follows from Lemma 4.3.

Suppose that (c) holds. Let η be an F -weight. Because $w^{-1} \in W(F)$, the weight $w^{-1}\eta$ is an F -weight. We obtain $\eta = ww^{-1}\eta \leq w^{-1}\eta$ by (c). From the equivalence of (a) and (b) we get $w^{-1} \in W_*(F)$, from which (a) follows. \square

Now we can show:

Theorem 4.39 *Let $\epsilon, \delta \in \{+, -\}$. Then*

$$M = \bigsqcup_{x \in R} B^\epsilon x B^\delta.$$

Proof. Every idempotent $e \in E(\overline{T})$ can be written uniquely in the form $e = \sigma f \sigma^{-1}$ with $f \in \Lambda$ and $\sigma \in W^{\lambda(f)}$. Moreover, $W^{\lambda(f)} = \{w \in W \mid w\alpha \in \Delta_+^{re} \text{ for all } \alpha \in \lambda(f)\}$ by [1, Proposition 2.20] and [10, Lemma 3.11 a)]. From Theorem 4.37 we obtain

$$M = \bigsqcup_{e \in E(\overline{T})} G e B^\delta = \bigsqcup_{\sigma \in W^{\lambda(f)}, f \in \Lambda} G f \sigma^{-1} B^\delta. \quad (36)$$

Now we consider a set $G f \sigma^{-1} B^\delta$ of this union. Inserting $G = \bigsqcup_{w \in W^{\lambda(f)}} B^\epsilon w P_{\lambda(f)}$ and the decomposition

$$P_{\lambda(f)} = T G_{\lambda^*(f)} G_{\lambda_*(f)} U^{\lambda(f)} = U_{\lambda^*(f)}^\epsilon T N_{\lambda^*(f)} U_{\lambda^*(f)}^\delta G_{\lambda_*(f)} U^{\lambda(f)},$$

we obtain from Corollaries 4.14 and 4.13 that

$$\begin{aligned} G f \sigma^{-1} B^\delta &= \bigcup_{w \in W^{\lambda(f)}} B^\epsilon w P_{\lambda(f)} f \sigma^{-1} B^\delta = \bigcup_{w \in W^{\lambda(f)}} B^\epsilon w U_{\lambda^*(f)}^\epsilon W_{\lambda^*(f)} U_{\lambda^*(f)}^\delta f \sigma^{-1} B^\delta, \\ &= \bigcup_{w \in W^{\lambda(f)}} B^\epsilon w U_{\lambda^*(f)}^\epsilon w^{-1} w W_{\lambda^*(f)} f \sigma^{-1} \sigma U_{\lambda^*(f)}^\delta \sigma^{-1} B^\delta. \end{aligned}$$

Here, $wU_{\lambda^*(f)}^\epsilon w^{-1} \subseteq B^\epsilon$ and $\sigma U_{\lambda^*(f)}^\delta \sigma^{-1} \subseteq B^\delta$. In view of $f = W_{\lambda^*(f)} f$ we get

$$Gf\sigma^{-1}B^\delta = \bigcup_{w \in W^{\lambda(f)}} B^\epsilon w W_{\lambda(f)} f \sigma^{-1} B^\delta = \bigcup_{w \in W} B^\epsilon w \sigma^{-1} e B^\delta = \bigcup_{x \in W e} B^\epsilon x B^\delta. \quad (37)$$

We next show that this union is disjoint. Note that $e = e(F)$ for a face F of the weight hull H . If $B^\epsilon w e B^\delta \cap B^\epsilon \tilde{w} e B^\delta \neq \emptyset$ there exist $u_\epsilon \in U^\epsilon$, $\tilde{u}_\delta \in U^\delta$ and $n, \tilde{n} \in N$ projecting to w, \tilde{w} , respectively, such that $u_\epsilon \tilde{n} e(F) \tilde{u}_\delta = n e(F)$. Therefore,

$$(n^{-1} u_\epsilon n)(n^{-1} \tilde{n}) e(F) = e(F) \tilde{u}_\delta^{-1}. \quad (38)$$

Comparing the images of both sides in (38), we get $(n^{-1} u_\epsilon n) V_{w^{-1} \tilde{w} F} = V_F$, and equivalently $V_{w^{-1} \tilde{w} F} = (n^{-1} u_\epsilon^{-1} n) V_F$. By the action of $n^{-1} U^\epsilon n$ on the weight spaces we conclude that $w^{-1} \tilde{w}(F \cap P(V)) \subseteq F \cap P(V)$ and $w^{-1} \tilde{w}(F \cap P(V)) \supseteq F \cap P(V)$. From Lemma 4.3 (a) we obtain $w^{-1} \tilde{w} \in W(F)$.

Let η be an F -weight. Evaluating both sides of (38) at $v_\eta \in V_\eta \setminus \{0\}$, we find

$$w^{-1} \tilde{w} \eta \in \text{supp}((n^{-1} u_\epsilon n)(n^{-1} \tilde{n}) v_\eta) = \text{supp}(e(F) \tilde{u}_\delta^{-1} v_\eta) \subseteq \eta + Q_\delta.$$

If δ is equal to $+$, we get $w^{-1} \tilde{w} \eta \geq \eta$ for all F -weights η . If δ is equal to $-$, we have $w^{-1} \tilde{w} \eta \leq \eta$ for all F -weights η . From Lemma 4.38 we find $w^{-1} \tilde{w} \in W_*(F)$ in both cases. Hence, $w e(F) = \tilde{w} e(F)$ by Theorem 4.32.

Inserting the disjoint union (37) in (36) we obtain

$$M = \bigsqcup_{w \in W, e \in E(\overline{T})} B^\epsilon w e B^\delta = \bigsqcup_{x \in R} B^\epsilon x B^\delta.$$

□

In the following lemma we generalize some properties of the twin BN-pairs (B^\pm, N) of G given in [1, Definition 6.55, Lemma 6.80] to the monoid M .

Lemma 4.40 *Let $\alpha \in \Pi$, $x \in R$, and $\epsilon, \delta \in \{+, -\}$. Then*

- (a) $(B^\epsilon r_\alpha B^\epsilon)(B^\epsilon x B^\delta) \subseteq B^\epsilon r_\alpha x B^\delta \cup B^\epsilon x B^\delta$.
- (b) $(B^\delta x B^\epsilon)(B^\epsilon r_\alpha B^\epsilon) \subseteq B^\delta x r_\alpha B^\epsilon \cup B^\delta x B^\epsilon$.

Proof. It suffices to show (a). Then (b) follows from (a) by applying the Chevalley anti-involution. We write x in the form $x = we$ with $w \in W$ and $e \in E(R)$. Recall that

$$\Delta^{re} = \Delta_p(e) \cup \Delta_*(e) \cup \Delta^*(e) \cup \Delta_n(e).$$

From [10, Proposition 4.2] we get $\mathbf{U} = \mathbf{U}_\alpha \mathbf{U}^\alpha = \mathbf{U}^\alpha \mathbf{U}_\alpha$ where $\mathbf{U}^\alpha := \mathbf{U} \cap r_\alpha \mathbf{U} r_\alpha$. Applying ρ , applying the Chevalley anti-involution, we find

$$U^\epsilon = U^{\epsilon\alpha} U_{\epsilon\alpha} \quad \text{where} \quad U^{\epsilon\alpha} := U^\epsilon \cap r_\alpha U^\epsilon r_\alpha.$$

If $\epsilon w^{-1} \alpha \in \Delta_p(e) \cup \Delta_*(e) \cup (\Delta^*(e) \cap \Delta_\delta^{re})$ then by Corollary 4.15 we obtain

$$r_\alpha B^\epsilon w e = r_\alpha U^{\epsilon\alpha} U_{\epsilon\alpha} w T e = U^{\epsilon\alpha} r_\alpha U_{\epsilon\alpha} w T e = U^{\epsilon\alpha} r_\alpha w T U_{\epsilon w^{-1} \alpha} e \subseteq B^\epsilon r_\alpha w e B^\delta.$$

Hence $B^\epsilon r_\alpha B^\epsilon w e B^\delta = B^\epsilon r_\alpha w e B^\delta$.

We have $r_\alpha B^\epsilon r_\alpha \subseteq B^\epsilon \cup B^\epsilon r_\alpha B^\epsilon$ because (B^ϵ, N) is a BN-pair of G . If $\epsilon w^{-1} \alpha \in (\Delta^*(e) \cap \Delta_{-\delta}^{r_\epsilon}) \cup \Delta_n(e)$ then $\epsilon(r_\alpha w)^{-1} \alpha = -\epsilon w^{-1} \alpha \in \Delta_p(e) \cup (\Delta^*(e) \cap \Delta_\delta^{r_\epsilon})$, and we find

$$\begin{aligned} B^\epsilon r_\alpha B^\epsilon w e B^\delta &= B^\epsilon r_\alpha B^\epsilon r_\alpha r_\alpha w e B^\delta \subseteq B^\epsilon (B^\epsilon \cup B^\epsilon r_\alpha B^\epsilon) r_\alpha w e B^\delta \\ &= B^\epsilon r_\alpha w e B^\delta \cup B^\epsilon r_\alpha B^\epsilon r_\alpha w e B^\delta = B^\epsilon r_\alpha w e B^\delta \cup B^\epsilon r_\alpha r_\alpha w e B^\delta \\ &= B^\epsilon r_\alpha w e B^\delta \cup B^\epsilon w e B^\delta. \end{aligned}$$

□

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